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THE UNIVERSITY OF ALBERTA

PARTICLE-LIKE SOLUTIONS OF NON-LINEAR SIX-VECTOR FIELD EQUATIONS

A DISSERTATION
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
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by
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ABSTRACT

A brief review is given of attempts to formulate a unitary field theory involving the electromagnetic field. This thesis concerns itself with another such attempt, based on a proposal by Dr. H. Schiff for non-linear six-vector field equations.

The static case of these equations has been considered and analytic solutions were obtained for two special forms of the static equation. A third form of the static equation was solved on the LFP-30 computer and studied in some detail. It is shown that the latter solutions describe particles with no charge.

For the third form of the equation, three discrete solutions were found on the computer. Using a heuristic expression for the total energy of these particles, a value was obtained for the coupling constant of this field which was of the order of the coupling constant for "strong" interactions.

Interaction potentials between the particles were evaluated, using analytic approximations to the computer solutions. A ratio of the energy of the first particle to that of a compound particle composed of the first two particles was found to be comparable to the ratio of pi-meson to nucleon mass.

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I. INTRODUCTION

Maxwell's classical electromagnetic equations are dualistic in form. To describe the physical world two separate entities are needed: fields and particles. The particles form the sources for the field and interact with the field but are not a part of the field. They are assumed to be point particles with their only characteristics being charge and mass. The dualistic nature of the theory is clearly exhibited by the Lagrangian for a particle field system:[1]

$$S = -\int mc\,ds - \frac{i}{c^2}\int A_i j_i\,d\tau + \frac{i}{16\pi c}\int F_{ik}^2\,d\tau \tag{1.1}$$

in which the field, the particles and the interaction are introduced separately.*

The electromagnetic field tensor F_{ik} is defined as the curl of a four-vector potential, A_i, in the following manner:

$$F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \tag{1.2}$$

Because in the classical theory the four-potential is introduced essentially as a mathematical artifice, one

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[1] See, for example, Landau and Lifschitz, THE CLASSICAL THEORY OF FIELDS, (Addison-Wesley Press, Inc.) p. 77.

* See Appendix for definitions of symbols.

of the requirements which has been imposed on the classical theory is that of gauge invariance, i.e. the equations are to be invariant under the transformation,

$$A_i' = A_i + \frac{\partial S}{\partial x_i} \tag{1.3}$$

where S is some arbitrary scalar function.

The total momentum and energy of the electromagnetic field can be expressed as the integrals of a tensor T_{ik} called the energy-momentum tensor. These expressions are:[2]

$$P_i = -\frac{i}{c}\int T_{ik}\, dS_k$$

$$T_{ik} = \frac{1}{4\pi}\left(F_{il}F_{kl} - \frac{1}{4}F_{lm}^2\,\delta_{ik}\right)$$

$$P_{1,2,3} = P_{x,y,z}\,; \qquad P_4 = \frac{i\mathcal{E}}{c} \tag{1.4}$$

The integration is extended over a hypersurface containing all of three dimensional space. If the integration is carried out on the hyperplane t = a constant, we then have:

$$P_\mu = -\frac{i}{c}\int T_{4\mu}\, dV$$

$$\mathcal{E} = -\int T_{44}\, dV \tag{1.5}$$

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[2] Landau and Lifschitz, loc. cit., P. 82-86.

Because of the dualistic nature of the theory, profound unsolved problems exist which emphasize the need for a more satisfactory theory. Some of these difficulties are considered below.

In the spherically symmetric static case, the electric field for the electron follows the well-known Coulomb's law,

$$E_r = -\frac{e}{r^2} \tag{1.6}$$

Using eqns. (1.4), (1.5) and (1.6), we obtain the following expression for the total energy of the electron.

$$\mathcal{E} = \frac{1}{8\pi}\int \frac{e^2}{r^4}\, dV \tag{1.7}$$

If the electron is a point particle, the integral must be taken up to the origin and the total energy becomes infinite.

Lorentz and Abraham removed this difficulty by assuming a finite size for the electron. If the electron is assumed to be a sphere of radius "a", the integral can be evaluated and the total energy becomes,

$$\mathcal{E} = \frac{e^2}{2a} \tag{1.8}$$

This theory introduces the necessity for some cohesive force to hold the electron together, which must be nonelectromagnetic in origin.

Another difficulty with the finite electron in the classical theory is that the four-momentum P_i does not obey the relativistic transformation laws for a four-vector. We consider an electron moving with a velicity "v" along the z-axis. Then using eqn. (1.5), we have that

$$\mathcal{E} = -\int T_{44}\, dV$$

$$P_z = 0 \tag{1.9}$$

in the rest frame of the electron. In the frame of the observer, we have

$$\mathcal{E}' = -\int T_{44}'\, dV'$$

$$P_z' = -\frac{i}{c}\int T_{34}'\, dV' \tag{1.10}$$

Since in the rest frame of the electron T_{34} is zero, the result of using the relativistic transformation formulas[3] is

$$T_{44}' = \frac{T_{44} - \frac{v^2}{c^2} T_{33}}{1 - \frac{v^2}{c^2}}$$

$$T_{34}' = \frac{-\frac{iv}{c}\left(T_{44} - T_{33}\right)}{1 - \frac{v^2}{c^2}}$$

$$dV' = \sqrt{1 - \frac{v^2}{c^2}}\; dV \tag{1.11}$$

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[3] Landau and Lifschitz, loc. cit., p. 18.

Substitution of the above expressions into eqn. (1.10) gives:

$$\mathcal{E}' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[-\int T_{44}\,dV + \frac{v^2}{c^2}\int T_{33}\,dV\right]$$

$$P_z' = \frac{\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\left[-\int T_{44}\,dV + \int T_{33}\,dV\right] \qquad (1.12)$$

Since P_i is a four-vector, a Lorentz transformation must give:

$$\mathcal{E}' = \frac{\mathcal{E}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\int - T_{44}\,dV$$

$$P_z' = \frac{\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\int - T_{44}\,dV \qquad (1.13)$$

If eqns. (1.12) and (1.13) are to be equivalent, the following equation must be satisfied.

$$\int T_{33}\,dV = 0 \qquad (1.14)$$

By a similar argument, one can show that the integrals of T_{11} and T_{22} in the rest frame of the electron must also be equal to zero.

For an electron with a finite radius "a", these integrals are not equal to zero. Their values are:

$$\int T_{11}\,dV = \int T_{22}\,dV = \int T_{33}\,dV = -\frac{1}{6}\frac{e^2}{a} \tag{1.15}$$

Thus the finite electron does not have the proper transformation properties.

Aside from the "dualistic" difficulties, one also recognizes limitations of the classical theory vis-a-vis quantum processes. One example of this is the phenomenon of photon-photon scattering. Since Maxwell's equations are linear, the principle of superposition holds. Thus in the classical theory, one electromagnetic wave could not interact with another one and be scattered by it.

Considerations of this sort have led to attempts to abandon the dualistic viewpoint and adopt a unitarian one, in which the electromagnetic field and matter are considered to be different states of the same physical entity.

In this thesis we will examine attempts to formulate a unitary theory in which matter is considered as essentially electromagnetic in nature. This leads naturally to a non-linear generalization of Maxwell's equations.

In Section II the theory of Mie is outlined briefly and its limitations are indicated. In Section III we consider the theory of Born which attempts to overcome some of the difficulties inherent in Mie's theory but which has unsatisfactory features of its own. In Section IV we give an example of another attempt by Rosen, which requires the introduction of a subsidiary field. In Section V we examine in some detail suggestions for a unitary theory proposed by Dr. H. Schiff in which the quantum mechanical field equations provide a point of departure for the construction of a non-linear six-vector theory. Finally in Section VI certain tentative conclusions are drawn with regard to this latter theory.

II. MIE'S THEORY

The first attempts to obtain a non-linear generalization for the electromagnetic field equations were made by Gustav Mie[4]. Mie assumed that the electromagnetic field is the fundamental entity and that matter is just a manifestation of this field. In this picture the electron is not a foreign body in the electromagnetic field but a special form of the field. It is not a point charge but a distributed charge with most of the charge concentrated at the center. The principle of relativity was also assumed to be valid.

Mie describes the electromagnetic field entirely in terms of the so-called entities of quantity; $\vec{D}$, $\vec{H}$, ρ, and $\vec{J}$; and the entities of intensity; $\vec{E}$, $\vec{H}$, φ, and $\vec{A}$. These entities obey the equations:

$$\text{curl}\ \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi \vec{J}}{c} \tag{2.1}$$

$$\text{div}\ \vec{D} = 4\pi\rho \tag{2.2}$$

$$\vec{B} = \text{curl}\ \vec{A} \tag{2.3}$$

$$\vec{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{2.4}$$

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[4] Annalen der Physik, Vol 37, p. 511 (1912); Vol. 39, p. 1, (1912).

These equations can be put into four-dimensional notation. The entities, F_{ik}, P_{ik}, j_i, and A_i are defined as:

$$F_{ik} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix} \tag{2.5}$$

$$P_{ik} = \begin{pmatrix} 0 & H_z & -H_y & -iD_x \\ -H_z & 0 & H_x & -iD_y \\ H_y & -iH_x & 0 & -iD_z \\ iD_x & iD_y & iD_z & 0 \end{pmatrix} \tag{2.6}$$

$$x_{1,2,3} = x, y, z \quad ; \quad x_4 = ict \tag{2.7}$$

$$j_{1,2,3} = J_{x,y,z} \quad ; \quad j_4 = ic\rho \tag{2.8}$$

$$A_{1,2,3} = A_{x,y,z} \quad ; \quad A_4 = i\varphi \tag{2.9}$$

The Latin indices take on the values from one to four, and all repeated indices are summed from one to four. With this notation eqns. (2.1) - (2.4) can be written,

$$\frac{\partial P_{ik}}{\partial x_k} = \frac{4\pi j_i}{c} \tag{2.10}$$

$$F_{ih} = \frac{\partial A_h}{\partial x_i} - \frac{\partial A_i}{\partial x_h} \qquad (2.11)$$

The entities $\vec{E}$ and $\vec{B}$ are the electric and magnetic intensities, respectively, and $\vec{D}$ and $\vec{H}$ are the electric and magnetic field strengths. The entities $\vec{A}$ and φ are the vector and scalar potentials, and $\vec{J}$ and ρ are the current and charge densities.

Two more equations follow from eqn. (2.10) and (2.11). These are

$$\frac{\partial j_i}{\partial x_i} = 0 \qquad (2.12)$$

$$\frac{\partial F_{ih}}{\partial x_l} + \frac{\partial F_{hl}}{\partial x_i} + \frac{\partial F_{li}}{\partial x_h} = 0 \qquad (2.13)$$

These equations can be written in three dimensional notation as:

$$\operatorname{div} \vec{J} + \frac{\partial \rho}{\partial t} \qquad (2.12a)$$

$$\operatorname{div} \vec{B} = 0 \qquad (2.13a)$$

$$\operatorname{curl} \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \qquad (2.13b)$$

The entities of quantity, P_{ik} and j_i, are complex functions of the entities of intensity, F_{ik} and A_i. This relation is found through a variational principle. Mie considers a "world function" or Lagrangian which is a function of the entities of intensity. Then the equations of motion are such as to make the space-time integral of this Lagrangian a minimum.

$$\delta \int \mathcal{L}(F_{ik}, A_i)\, d\tau = 0 \qquad d\tau = dx\,dy\,dz\,dict$$

$$\int \left(\frac{1}{2} \frac{\partial \mathcal{L}}{\partial F_{ik}} \delta F_{ik} + \frac{\partial \mathcal{L}}{\partial A_i} \delta A_i \right) d\tau = 0$$

$$\int \left[-\frac{1}{2} \frac{\partial \mathcal{L}}{\partial F_{ik}} \left(\frac{\partial\, \delta A_i}{\partial x_k} - \frac{\partial\, \delta A_k}{\partial x_i} \right) + \frac{\partial \mathcal{L}}{\partial A_i} \delta A_i \right] d\tau = 0$$

$$\int \left(- \frac{\partial \mathcal{L}}{\partial F_{ik}} \frac{\partial\, \delta A_i}{\partial x_k} + \frac{\partial \mathcal{L}}{\partial A_i} \delta A_i \right) d\tau = 0$$

If the fields go to zero at large distances, the divergence theorem can be used to transform this integral to the form:

$$\int \left(\frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial F_{ik}} + \frac{\partial \mathcal{L}}{\partial A_i} \right) \delta A_i\, d\tau = 0$$

Since the variation of the A_i is arbitrary, the integrand must be identically zero. Thus we have

$$\frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial F_{ik}} + \frac{\partial \mathcal{L}}{\partial A_i} = 0 \qquad (2.14)$$

If this equation is to correspond to eqn. (2.10), the following relationship between the entities of quantity and the entities of intensity must be satisfied:

$$P_{ik} = -4\pi \frac{\partial \mathcal{L}}{\partial F_{ik}} \qquad (2.15)$$

$$\frac{j_i}{c} = \frac{\partial \mathcal{L}}{\partial A_i} \qquad (2.16)$$

Of course the relationship is maintained if the right hand sides of the eqns. (2.15) and (2.16) are multiplied by an arbitrary constant. The constant is chosen so that for Maxwell's Lagrangian, $\mathcal{L} = -\frac{1}{16\pi} F_{ik}^2$, we have $P_{ik} = F_{ik}$.

In three dimensional notation these equations become:

$$\vec{H} = -4\pi \frac{\partial \mathcal{L}}{\partial \vec{B}}$$

$$\vec{D} = 4\pi \frac{\partial \mathcal{L}}{\partial \vec{E}} \qquad (2.15a)$$

$$\frac{\vec{J}}{c} = \frac{\partial \mathcal{L}}{\partial \vec{A}} \qquad (2.16a)$$

$$\rho = -\frac{\partial \mathcal{L}}{\partial \varphi}$$

where

$$\frac{\partial \mathcal{L}}{\partial \vec{B}} = \hat{i}\,\frac{\partial \mathcal{L}}{\partial B_x} + \hat{j}\,\frac{\partial \mathcal{L}}{\partial B_y} + \hat{k}\,\frac{\partial \mathcal{L}}{\partial B_z}$$

The entities of intensity can also be considered as complex functions of the entities of quantity. This relationship is determined through a Hamiltonian which is a function of the entities of quantity. The Hamiltonian, $\mathcal{H}$, is defined by Mie as:

$$\mathcal{H} = -\frac{1}{8\pi} F_{ik} P_{ik} + \frac{1}{c} A_i j_i - \mathcal{L} \qquad (2.17)$$

Using eqns. (2.15) and (2.16), we have,

$$d\mathcal{H} = -\frac{1}{8\pi} F_{ik}\, dP_{ik} + \frac{A_i}{c}\, dj_i$$

Using the chain rule for differentials, we obtain the following equations.

$$\frac{\partial \mathcal{H}}{\partial P_{ik}} = -\frac{1}{4\pi} F_{ik} \qquad (2.18)$$

$$\frac{\partial \mathcal{H}}{\partial j_i} = \frac{A_i}{c} \qquad (2.19)$$

In three dimensional notation these equations become:

$$\vec{B} = -4\pi \frac{\partial \mathcal{H}}{\partial \vec{H}}$$

$$\vec{E} = 4\pi \frac{\partial \mathcal{H}}{\partial \vec{D}} \qquad (2.18a)$$

$$\frac{\vec{A}}{c} = \frac{\partial \mathcal{H}}{\partial \vec{J}}$$

$$\varphi = -\frac{\partial \mathcal{H}}{\partial \rho} \qquad (2.19a)$$

By using a variational principle on the Lagrangian, we obtain eqn. (2.10) from eqn. (2.11). We can reverse this procedure by carrying out a variation of the Hamiltonian, starting with eqn. (2.10), and using eqns. (2.18) and (2.19) as definitions.

The variational principle is :

$$\delta \int \mathcal{H} d\tau = 0$$

$$\int \left(\frac{1}{2} \frac{\partial \mathcal{H}}{\partial P_{ih}} \delta P_{ih} + \frac{\partial \mathcal{H}}{\partial j_i} \delta j_i \right) d\tau = 0$$

Using eqn. (2.10), we have

$$\delta j_i = \frac{c}{4\pi} \frac{\partial}{\partial x_h} \delta P_{ih}$$

$$\int \left(-\frac{1}{2} F_{ih} \delta P_{ih} - A_i \frac{\partial}{\partial x_h} \delta P_{ih} \right) d\tau = 0$$

The divergence theorem gives:

$$\int \left(-\frac{1}{2} F_{ih} \, \delta P_{ih} - \frac{\partial A_i}{\partial x_h} \, \delta P_{ih} \right) d\tau = 0$$

$$\frac{\partial A_i}{\partial x_h} \, \delta P_{ih} = \frac{1}{2} \left(\frac{\partial A_i}{\partial x_h} - \frac{\partial A_h}{\partial x_i} \right) \delta P_{ih}$$

Therefore,

$$\int \left[-\frac{1}{2} F_{ih} - \frac{1}{2} \left(\frac{\partial A_i}{\partial x_h} - \frac{\partial A_h}{\partial x_i} \right) \right] d\tau = 0$$

and

$$F_{ih} = \frac{\partial A_h}{\partial x_i} - \frac{\partial A_i}{\partial x_h}$$

Thus minimizing the space-time integral of the Hamiltonian leads to eqn. (2.11).

The problem of formulating a unitary field theory has thus been reduced to one of determining the correct form of the Lagrangian or Hamiltonian. Although their actual form is not known, the principle of relativity requires that they be four dimensional scalars or invariants. Therefore, they must be functions of the invariants which can be constructed from the field entities.

It can be shown that there are only four independent invariants which can be constructed from the entities

of intensity, to be used in the Lagrangian. These four are: F^2_{ik}, $F_{ik}F^*_{ik}$, A_i^2, and $(F_{ik}A_k)^2$. The dual tensor, F^*_{ik}, is defined as $\frac{1}{2}e_{iklm}F_{lm}$, where e_{iklm} is the completely antisymmetric pseudo-tensor of the fourth rank.

$$e_{iklm} = \begin{cases} 1 & iklm \text{ an even permutation of } 1234 \\ -1 & iklm \text{ an odd permutation of } 1234 \\ 0 & \text{any two indices the same} \end{cases} \tag{2.20}$$

Therefore

$$F_{ik}^* = \begin{pmatrix} 0 & -iE_z & iE_y & B_x \\ iE_z & 0 & -iE_x & B_y \\ -iE_y & iE_x & 0 & B_z \\ -B_x & -B_y & -B_z & 0 \end{pmatrix} \tag{2.21}$$

The invariants in three dimensional notation are:

$$\begin{aligned} F_{ik}^2 &= 2(B^2 - E^2) \\ F_{ik}F_{ik}^* &= -4i(\vec{B}\cdot\vec{E}) \\ A_i^2 &= A^2 - \varphi^2 \\ (F_{ik}A_k)^2 &= (\vec{A}\times\vec{B} + \varphi\vec{E})^2 - (\vec{A}\cdot\vec{E})^2 \end{aligned} \tag{2.22}$$

Similarly, there are only four independent invariants which can be constructed from the entities of quantity. These are:

$$P_{ih}^2 = 2(H^2 - D^2)$$

$$P_{ih} P_{ih}^* = -4i(\vec{H}\cdot\vec{D})$$

$$j_i^2 = J^2 - c^2\rho^2 \qquad (2.23)$$

$$(P_{ih} j_h)^2 = (\vec{J}\times\vec{H} + c\rho\vec{D})^2 - (\vec{J}\cdot\vec{D})^2$$

A conservation law can be drrived using the field equations (2.10) and (2.13). If the Lagrangian is differentiated with respect to the coordinates, the result is:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{1}{2}\frac{\partial \mathcal{L}}{\partial F_{hl}}\frac{\partial F_{hl}}{\partial x_i} + \frac{\partial \mathcal{L}}{\partial A_l}\frac{\partial A_l}{\partial x_i}$$

Using eqns. (2.15) and (2.16) we have,

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{1}{8\pi} P_{hl}\left(\frac{\partial F_{hl}}{\partial x_i}\right) + \frac{j_l}{c}\frac{\partial A_l}{\partial x_i}$$

Uwing eqn. (2.13),

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{1}{8\pi} P_{hl}\left(\frac{\partial F_{li}}{\partial x_h} + \frac{\partial F_{ih}}{\partial x_l}\right) + \frac{j_l}{c}\frac{\partial A_l}{\partial x_i}$$

$$= \frac{1}{4\pi} P_{hl}\frac{\partial F_{li}}{\partial x_h} + \frac{j_l}{c}\frac{\partial A_l}{\partial x_i}$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial x_h}(P_{hl}F_{li}) - \frac{F_{li}}{4\pi}\frac{\partial P_{hl}}{\partial x_h} + \frac{j_l}{c}\frac{\partial A_l}{\partial x_i}$$

Using eqns. (2.10) and (2.11),

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{1}{4\pi}\frac{\partial}{\partial x_h}(F_{il}P_{hl}) + \left(\frac{\partial A_i}{\partial x_l} - \frac{\partial A_l}{\partial x_i}\right)\frac{j_l}{c} + \frac{j_l}{c}\frac{\partial A_l}{\partial x_i}$$

$$= -\frac{1}{4\pi}\frac{\partial}{\partial x_h}(F_{il}P_{hl}) + \frac{j_h}{c}\frac{\partial A_i}{\partial x_h}$$

Using eqn. (2.12),

$$\frac{\partial}{\partial x_h}\delta_{ih}\mathcal{L} = -\frac{1}{4\pi}\frac{\partial}{\partial x_h}(F_{il}P_{hl}) + \frac{\partial}{\partial x_h}\frac{A_i j_h}{c}$$

where δ_{ih} is the Kronecker delta. If the tensor T_{ik} is defined as:

$$T_{ih} = \delta_{ih}\mathcal{L} + \frac{1}{4\pi}F_{il}P_{hl} - \frac{A_i j_h}{c} \qquad (2.24)$$

Then $$\frac{\partial}{\partial x_h}T_{ih} = 0$$

Therefore the integral of this tensor over a hypersurface containing all of three dimensional space is conserved. This tensor is called the energy-momentum tensor. The total momentum or energy of the field can be found by using eqn. (1.4).

Although Mie has developed a general form for the electromagnetic field equations through the use of a Lagrangian, he gives no way of determining the form of this Lagrangian. His only criterian are that it be a relativistic invariant and that at large distances from the center of the electron his Lagrangian should approach Maxwell's Lagrangian.

Mie states that since the results of applying his theory would be compared with experiments for the static case, he uses only the invariants F_{ik}^2 and A_i^2, since in the static case $F_{ik}F^*_{ik}$ is equal to zero and $(F_{ik}A_k)^2$ can be expressed as the product of the first two invariants. He has considered a Langrangian of the form:

$$\mathcal{L} = -\frac{1}{16\pi} F_{ik}^2 + a\,|A_i^2|^n$$

Then,

$$P_{ik} = F_{ik}$$

$$j_i = 2nac\,A_i\,|A_k^2|^{n-1}$$

He has given the solutions for n = 3. He has shown that in this case, one obtains a highly concentrated charge distribution, but it is unstable in the field of another electron.

The use of the invariant A_i^2 in Mie's theory, introduces the potentials directly into the field equations. Therefore the equations of motion are not gauge invariant, i.e. his theory depends on the absolute value of the potentials. Thus in the static case, if the external potential were changed by a constant, the form of the solution could change. Also since the equations are derived on a classical basis, there is no possibility of explaining the spin of the electron.

Despite its shortcomings, Mie's general formalism has been the starting point for most unitary field theories, such as the one by Born, which we consider in the next section.

III. BORN'S THEORY OF THE ELECTRON

In 1934, Born[5], in colloboration with Infeld, developed a Lagrangian to describe the electron. Their theory was gauge invariant in that only the fields and not the potentials appear explicitly in the Lagrangian. They developed a Lagrangian such that the action integral, $\int \mathcal{L}\, d\tau$, was an invariant under the transformations of general relativity.

Using a covariant tensor a_{kl}, the simplest form of the Lagrangian so that the action integral is invariant is the determinant $\sqrt{|a_{kl}|}$. The tensor a_{kl} is split into a symmetrical and an antisymmetrical part.

$$a_{kl} = g_{kl} + f_{kl} \quad ; \qquad g_{kl} = g_{lk} \, , \; f_{kl} = -f_{lk}$$

The symmetrical part is identified with the metric tensor and the antisymmetrical part with the electromagnetic field tensor.

The simplest Lagrangian which can be constructed with these entities is:

$$\mathcal{L} = \alpha \sqrt{- |g_{kl} + f_{kl}|} + \beta \sqrt{-|g_{kl}|} + \gamma \sqrt{|f_{kl}}$$

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[5] Proc. Roy. Soc. London, A144, p. 425, 1934.

The last term can be neglected since it gives no contribution to the field equations. In cartesian coordinates and with the notation of special relativity, we have,

$$\mathcal{L} = -\frac{b^2}{4\pi}\left(\sqrt{1+F-G^2} - 1\right) \tag{3.1}$$

$$F = \frac{F_{ik}^2}{2b^2} = \frac{B^2 - E^2}{b^2} \tag{3.2}$$

$$G = \frac{i}{4b^2}\left(F_{ik}F_{ik}^*\right) = \frac{1}{b^2}\vec{E}\cdot\vec{B} \tag{3.3}$$

The constants are chosen so that in the limit of weak fields, this Lagrangian reduces to Maxwell's Lagrangian. Here "b" is a constant with the dimensions of field strength.

This Lagrangian sets an upper limit "b" on the strength of the fields, just as the transformations of special relativity set an upper limit "c" on the velocity of a particle.

The entities of quantity can be found by using eqns. (2.15) and (2.16). Thus we have,

$$P_{ik} = \frac{F_{ik} - iGF_{ik}^*}{\sqrt{1+F-G^2}} \tag{3.4}$$

$$j_i = 0 \tag{3.5}$$

$$\vec{H} = \frac{\vec{B} - \frac{1}{b^2}(\vec{B}\cdot\vec{E})\vec{E}}{\sqrt{1+F-G^2}} \tag{3.6}$$

$$\vec{D} = \frac{\vec{E} - \frac{1}{b^2}(\vec{B}\cdot\vec{E})\vec{B}}{\sqrt{1 + F - G^2}} \tag{3.7}$$

Then the equations of motion become:

$$\frac{\partial P_{ik}}{\partial x_k} = 0 \tag{3.8}$$

$$\operatorname{curl} \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = 0 \tag{3.9}$$

$$\operatorname{div} \vec{D} = 0 \tag{3.10}$$

By using eqn. (2.24), the energy-momentum tensor, T_{ik}, can be determined. For this Lagrangian, the energy-momentum tensor is:

$$T_{ik} = \frac{1}{4\pi}\left[\frac{F_{il}F_{kl} - b^2 G^2 \delta_{ik}}{\sqrt{1 + F - G^2}} - b^2\left(\sqrt{1+F-G^2} - 1\right)\delta_{ik}\right] \tag{3.11}$$

Born and Infeld have considered the static case with central symmetry. In this case the fields become:

$$\begin{gathered} \operatorname{div}\vec{D} = 0 \\ D_r = \frac{e}{r^2} \\ D_r = \frac{E_r}{\sqrt{1 - \frac{E_r^2}{b^2}}} \end{gathered} \tag{3.12}$$

$$E_r = \frac{\frac{e}{r^2}}{\sqrt{1+\frac{e^2}{b^2r^4}}} \qquad (3.13)$$

$$\varphi = \frac{e}{r_0} f\left(\frac{r}{r_0}\right) \; ; \; f(x) = \int_x^\infty \frac{dy}{\sqrt{1+y^4}} \; ; \; r_0 = \sqrt{\frac{e}{b}} \qquad (3.14)$$

The potential is finite and continuous everywhere and goes as e/r asymptotically. The value of the potential at the origin is $1.854e/r_0$. The electric field is also finite everywhere but it has a discontinuity in direction at the origin.

Born and Infeld consider the divergence of $\vec{E}$ as defining a spatial charge distribution, which they call the "free charge" density. The actual charge density is zero since the potentials do not appear explicitly in the Lagrangian. For the static case, this free charge density is:

$$\rho = \frac{e}{2\pi r_0^2 r\left[1+\left(\frac{r}{r_0}\right)^4\right]^{3/2}} \qquad (3.15)$$

It is evident that the charge density becomes very small when $r > r_0$. The total free charge over all of space is:

$$\int \rho \, dV = e$$

The value of the constant "b" in these equations is determined by calculating the total energy of this particle and equating this to the rest energy of the electron. For the static case we have,

$$T_{44} = -\frac{b^2}{4\pi}\left(\frac{1}{\sqrt{1-\frac{E^2}{b^2}}} - 1\right) \tag{3.16}$$

Using eqn. (1.5), we have,

$$\mathcal{E} = \int \frac{b^2}{4\pi}\left(\frac{1}{\sqrt{1-\frac{E^2}{b^2}}} - 1\right) dV$$

If eqn. (3.13) is substituted into the above, the result is:

$$\mathcal{E} = 1.2361\,\frac{e^2}{r_o} = mc^2 \tag{3.17}$$

where "m" is the rest mass of the electron. This gives

$$r_o = 2.28 \times 10^{-13}\ \text{cm.}$$

$$b = 9.18 \times 10^{15}\ \text{e.s.u.} \tag{3.18}$$

Because of the small value of the characteristic length of the electron and the large value of the maximum field

strength in this theory, experimental deviations from the classical laws would not be readily apparent.

Although Born's theory has the desirable property of gauge invariance, it is not truly unitary, since the definition of charge is artificial. Frankel[6] and Feenberg[7] have shown that the field equations must be supplemented with equations of motion for the electron for a complete description.

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[6] Proc. Roy. Soc. London, A146, p. 933, 1934.

[7] Phys. Rev., Vol. 47, p. 148, 1935.

IV. ROSEN'S FIELD THEORY

In this section, we indicate briefly, a somewhat different formulation due to Rosen.[8] He has shown that any Lagrangian which contains only the fields explicitly, has either a singular point or a discontinuity in direction of the field at the origin in the spherically symmetric static case. This requires that the field equations be supplemented with equations of motion for the particles. For a truly unitary theory, the potentials must appear directly in the Lagrangian. However, if this were the case, the equations would not be gauge invariant.

To remove this difficulty, Rosen introduced an additional scalar function which changes under a gauge transformation in such a manner that the total Lagrangian is gauge invariant. He chose a "matter" function, ψ , which transforms in the following manner:

$$\psi' = \psi e^{i \epsilon S}$$

as

$$A'_h = A_h + \frac{\partial S}{\partial x_h} \qquad (4.1)$$

Here, ϵ is a constant with the dimensions of inverse charge. Two simple functions of the potentials and this

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[8] Phys. Rev., Vol. 55, p. 94, 1939.

scalar function can be constructed, which are both Lorentz and gauge invariant. These are:

$$\psi \psi^* \tag{4.2}$$

$$\chi_k \chi_k^* \tag{4.3}$$

$$\chi_k = \frac{\partial \psi}{\partial x_k} - i \epsilon A_k \psi \tag{4.4}$$

where (*) stands for complex conjugate.

Rosen has chosen a Lagrangian of the form:

$$\mathcal{L} = - \frac{F_{ik}^2}{16\pi} - \chi_k \chi_k^* + \sigma^2 \psi \psi^* \tag{4.5}$$

where σ is a constant with dimensions of inverse length. Interestingly enough, about ten years previously, the same Lagrangian had actually been propsed by Gordon[9], Klein[10], and Schroedinger[11,12], although Rosen was apparently unaware of this.

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[9] Z. Phys., 40, 117, (1926)

[10] Z. Phys., 37, 895, (1926)

[11] Ann. der Phys., (4), 81 (1926); 82 (1927)

[12] Nature, Vol. 169, p. 538, (1952)

[13] Phys. Rev., Vol. 62, p. 436, (1942)

In using the variational principle, the potentials and the matter function are varied independently. This gives an additional set of equations besides the usual Maxwell's equations. For Rosen's Lagrangian the equations are:

$$\left(\frac{\partial}{\partial x_k} - i\epsilon A_k\right)^2 \psi + \sigma^2 \psi = 0 \qquad (4.6)$$

$$\frac{\partial F_{kl}}{\partial x_l} = \frac{4\pi}{c} j_k \qquad (4.7)$$

$$\frac{1}{c} j_k = \frac{1}{2} i\epsilon \left(\psi^* \chi_k - \psi \chi_k^*\right) \qquad (4.8)$$

Menius and Rosen[13] have solved the spherically symmetric static case numerically, and have been able to obtain electron-like solutions. The solutions however had the undesirable property of having the total energy negative.

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[13] Phys. Rev., Vol. 62, p. 436, (1942).

V. FOUNDATIONS OF A UNITARY FIELD THEORY

In this section we consider a proposal of Dr. H. Schiff for a unitary six-vector field theory.

Since the best description yet obtained for a particle is provided by the Dirac equation for the electron, it seems natural to use this equation as a basis for the development of such a theory, Dirac's equations can be written[14]

$$i \frac{\partial \phi}{\partial t} = m\phi + \vec{\sigma}\cdot\vec{p}\,\chi \tag{5.1}$$

$$i \frac{\partial \chi}{\partial t} = -m\chi + \vec{\sigma}\cdot\vec{p}\,\phi \tag{5.2}$$

with $\hbar = c = 1$.

The functions ϕ and χ are first rank spinors which under an inversion of the coordinate system transform as:

$$\phi' = \phi$$
$$\chi' = -\chi \tag{5.3}$$

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[14] Akhiezer and Berestetsky, Quantum Electrodynamics, (Consultants Bureau, Inc.) p. 55.

In eqns. (5.1) and (5.2), $\vec{p}$ is the momentum operator, -i grad, and $\vec{\sigma}$ is the set of Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.4}$$

If eqns. (5.1) and (5.2) are added and substracted the result is:

$$i \frac{\partial}{\partial t}(\phi+\chi) = m(\phi-\chi) + \vec{\sigma}\cdot\vec{p}\,(\phi+\chi) \tag{5.5}$$

$$i \frac{\partial}{\partial t}(\phi-\chi) = m(\phi+\chi - \vec{\sigma}\cdot\vec{p}\,(\phi-\chi) \tag{5.6}$$

Under spatial inversion, $(\phi+\chi)$ and $(\phi-\chi)$ transform into each other.

We now consider the complex field vector $\vec{F} = \vec{H} + i\vec{E}$, where $\vec{H}$ is an axial vector and $\vec{E}$ is a polar vector. It is known from the work of Laporte and Uhlenbeck[15] that appropriate linear combinations of the components of $\vec{F}$ transform as the components of a rank two spinor. Further by contraction, one can always form a first rank spinor by combining a second rank spinor with a first rank spinor. One can then show that if B is an arbitrary first

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[15] Laporte and Uhlenbeck, Phys. Rev., Vol. 37, p. 1380 (1931).

rank spinor, then $(\vec{\sigma}\cdot\vec{F})$B is also a first rank spinor. That $(\vec{\sigma}\cdot\vec{F})$B is a first rank spinor will be evident when we consider the two component form of the Dirac equation.

We now note that under inversion of corrdinates, $(\vec{\sigma}\cdot\vec{F})$ and $(\vec{\sigma}\cdot F^*)$ transform into each other. If we now choose two first rank spinors B and C which transform under inversion as $\phi+\chi$ and $\phi-\chi$ respectively, then it follows that $(\vec{\sigma}\cdot\vec{F})$B and $(\vec{\sigma}\cdot\vec{F}^*)$C also transform as $\phi+\chi$ and $\phi-\chi$ respectively.

Eqns. (5.5) and (5.6) can be easily combined to obtain a second order two component equation,

$$\left(\frac{\partial^2}{\partial x_h^2} - m^2\right)(\phi+\chi) = 0 \tag{5.7}$$

and a similar equation for $\phi-\chi$. In the presence of an electromagnetic field, this equation becomes:

$$\left[\left(i\frac{\partial}{\partial x_h} + eA_h\right)^2 + m^2 - e\vec{\sigma}\cdot\vec{F}\right](\phi+\chi) = 0 \tag{5.8}$$

It is now apparent that since eqn. (5.8) is covariant that $(\vec{\sigma}\cdot\vec{F})(\phi+\chi)$ must transform as $\phi+\chi$.

We now replace the spinor $\phi+\chi$ in eqn. (5.8) by $(\vec{\sigma}\cdot\vec{F})$B, with the A_k's in the equation being the potentials for $\vec{F}$. Then the equation becomes:

$$\left[\left(i\frac{\partial}{\partial x_h} + gA_h\right)^2 + m^2 - g\vec{\sigma}\cdot\vec{F}\right](\vec{\sigma}\cdot\vec{F})B = 0 \tag{5.9}$$

This is a relativistically invariant equation for a self interacting field of spin $\frac{1}{2}$. The constant "g" in the equation has the units of charge and gives a measure of the interaction.

Eqn. (5.9) suggests that for the boson case, we consider the following equation:

$$\left[\left(i\frac{\partial}{\partial x_h} + g A_h\right)^2 + m^2\right]\vec{F} = 0 \tag{5.10}$$

where the operator, $\left(i\frac{\partial}{\partial x_h} + g A_h\right)^2 + m^2$ is simply the Klein-Gordon operator with a self interaction.

If the constants, ħ and c, are restored, the equation becomes

$$\left[\left(i\hbar\frac{\partial}{\partial x_h} + \frac{g}{c} A_h\right)^2 + m^2c^2\right]\vec{F} = 0 \tag{5.11}$$

Equation (5.11) is not gauge invariant since the transformation

$$A_h' = A_h + \frac{\partial S}{\partial x_h}$$

gives

$$F' = F e^{i g \frac{S}{\hbar c}} \tag{5.12}$$

If we assume that H and E are real fields, eqn. (5.11) can be split into real and imaginary parts. Then the field equation can be written as:

$$\left[\frac{\partial^2}{\partial x_n^2} - \frac{g^2}{\hbar^2 c^2} A_n^2 - \frac{m^2 c^2}{\hbar^2}\right] F_{lm} + \frac{2g}{\hbar c} A_n \frac{\partial}{\partial x_n} i F_{lm}^{*} + \frac{g}{\hbar c} i F_{lm}^{*} \frac{\partial A_n}{\partial x_n} = 0 \tag{5.13}$$

where F_{lm} and F^*_{lm} are defined in eqns. (2.5) and (2.21). The static case of this equation is considered in the next section.

Attempts to find a Lagrangian for this theory have so far been unsuccessful. Presumably in order to obtain a Lagrangian which goes over into the Maxwell Lagrangian as "m" and "g" go to zero, one would require a first order boson equation. However since there are a number of different types of first order boson equations, all of which would lead to a Klein-Gordon type of second order equation, and also since we will restrict ourselves to the static case, we will use eqn. (5.11) and a heuristic expression for the energy.

VI. ANALYSIS OF THE STATIC CASE

For the static case and with $\vec{A} = 0$, eqn. (5.13) becomes

$$\nabla^2 \vec{E} - \frac{m^2c^2}{\hbar^2} \vec{E} + \frac{g^2}{\hbar^2c^2} \varphi^2 \vec{E} = 0$$

Putting $\vec{E} = -\text{grad}\ \varphi$ we get,

$$\nabla^2 \nabla \varphi - \frac{m^2c^2}{\hbar^2} \nabla \varphi + \frac{g^2}{\hbar^2c^2} \varphi^2 \nabla \varphi = 0$$

i.e.

$$\nabla\left(\nabla^2 \varphi - \frac{m^2c^2}{\hbar^2} \varphi + \frac{g^2}{\hbar^2c^2} \frac{\varphi^3}{3}\right) = 0$$

from which it follows that the expression in brackets is a constant (Υ , say), or,

$$\nabla^2 \varphi = \frac{m^2c^2}{\hbar^2} \varphi - \frac{g^2}{\hbar^2c^2} \frac{\varphi^3}{3} + \Upsilon \qquad (6.1)$$

It was not possible in general to obtain analytic solutions to this equation and numerical methods were necessary. However analytic solutions do exist for certain special cases, in particular the one dimensional case which we consider below.

The one dimensional form of eqn. (6.1) with the constant Υ set equal to zero is

$$\frac{d^2\varphi}{dx^2} = \frac{m^2c^2}{\hbar^2} \varphi - \frac{g^2}{\hbar^2c^2} \frac{\varphi^3}{3} \qquad (6.2)$$

The following substitutions are made into the equation:

$$\varphi = \frac{g\psi}{\lambda G} \qquad u = \frac{x}{\lambda}$$

$$\lambda = \frac{\hbar}{mc} \qquad G = \frac{g^2}{\hbar c} \tag{6.3}$$

The result is

$$\frac{d^2\psi}{dx^2} = \psi - \frac{\psi^3}{3} \tag{6.4}$$

where ψ and "u" are now dimensionless.

Integrating with respect to ψ , we obtain the first integral of eqn. (6.4), namely,

$$\left(\frac{d\psi}{du}\right)^2 = \psi^2 - \frac{\psi^4}{6} + \frac{C}{6} \tag{6.5}$$

This equation can be solved in terms of elliptic functions. The form of the solution depends on the constant of integration C. The solutions of this equation are

$$\text{(i)} \quad C = 0 , \qquad \psi = \frac{\sqrt{6}}{\cosh(u + u_0)} \tag{6.6}$$

$$\varphi = \frac{g}{\lambda G} \frac{\sqrt{6}}{\cosh\left(\frac{x + x_0}{\lambda}\right)}$$

(ii) $c > 0$, $k^2 = \dfrac{\sqrt{9+c}+3}{2\sqrt{9+c}}$

$$\varphi = \frac{g}{\lambda G}\,\frac{\sqrt{6}\,k}{\sqrt{2k^2-1}}\,\mathrm{cn}\left(\frac{x+x_0}{\lambda\sqrt{2k^2-1}}\right) \qquad (6.7)$$

(iii) $0 > c > -9$, $k^2 = \dfrac{2\sqrt{9+c}}{3+\sqrt{9+c}}$ (6.8)

$$\varphi = \frac{g}{\lambda G}\sqrt{\frac{6}{2-k^2}}\;\mathrm{dn}\left(\frac{x+x_0}{\lambda\sqrt{2-k^2}}\right.$$

(iv) $c < -9$, No real solution.

Here "k" is the modulus of the elliptic function. The constants "k" and "x_0" are arbitrary constants of integration.

All of these solutions are finite everywhere. The first solution has clearly decaying exponential asymptetic behaviour. The second and third solutions are periodic functions whose period and amplitude depend on the modulus "k".

The three dimensional case with spherical symmetry is now considered. Eqn. (6.1) then becomes

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\varphi}{dr} = \frac{m^2c^2}{\hbar^2}\varphi - \frac{g^2}{\hbar^2c^2}\frac{\varphi^3}{3} + \gamma \qquad (6.9)$$

If the following substitutions are made,

$$\varphi = \frac{g\psi}{\lambda G} \qquad r = \lambda x \qquad \gamma = \frac{g\beta}{\lambda^3 G} \qquad (6.10)$$

$$\lambda = \frac{\hbar}{mc} \qquad G = \frac{g^2}{\hbar c}$$

eqn. (6.9) becomes

$$\frac{1}{x^2}\frac{d}{dx}x^2\frac{d\psi}{dx} = \psi - \frac{\psi^3}{3} + \beta \tag{6.11}$$

Now the substitution,

$$\psi = A + \frac{f}{x} \tag{6.12}$$

is made, where "A" is a constant. The result is

$$\frac{d^2f}{dx^2} = \left(A - \frac{A^3}{3} + \beta\right) + (1 - A^2)f - \frac{Af^2}{x} - \frac{f^3}{3x^2}$$

The constant "A" is chosen so that the relation,

$$A - \frac{A^3}{3} + \beta = 0 \tag{6.13}$$

is satisfied. The resulting equation is

$$\frac{d^2f}{dx^2} = (1 - A^2)f - \frac{Af^2}{x} - \frac{f^3}{3x^2} \tag{6.14}$$

If the function "f" goes to zero sufficiently rapidly at large distances from the origin, the linear term in "f" on the right hand side of the equation becomes

the dominant term in the differential equation. Therefore if the constant "A" is less than one in absolute value, the function "f" goes as exp ($-\sqrt{1 - A^2}\, x$).

Only two cases of eqn. (6.14) have been investigated. These are $A = \pm 1$ and $A = 0$. The second case will be discussed more fully later. For the first case, eqn. (6.13) gives

$$\beta = \mp \frac{2}{3}$$

Eqn. (6.14) then becomes

$$\frac{d^2 f}{dx^2} = \mp \frac{f^2}{x} - \frac{f^3}{3x^2} \tag{6.15}$$

If the function "f" is written as a series of the inverse powers of "x" and substituted into the equation, an algebraic solution can be obtained by equating powers of "x^{-1}". This solution is

$$f = \mp \frac{12x}{6x^2+1}$$

which is finite at the origin and goes as the inverse power of "x" asymptotically. The function, ψ is thus

$$\psi = \pm \left(1 - \frac{12}{6x^2+1}\right) \tag{6.15}$$

and,

$$\varphi = \pm \frac{g}{\lambda G}\left(1 - \frac{12}{6\left(\frac{r}{\lambda}\right)^2+1}\right) \tag{6.17}$$

which is finite and non-zero at the origin and at infinity.

Solutions exist for $A^2 < 1$ and $A^2 = 1$, which have characteristically different asymptotic behaviours; exponential in the former and algebraic in the latter. This means that the value of the potential at infinity, "A", greatly influences the nature of the solutions. This clearly illustrates the importance of the choice of gauge in the original time dependent equation.

If A = 0 in eqn. (6.14), the equation becomes

$$\frac{d^2 f}{dx^2} = f - \frac{f^3}{3x^2} \qquad (6.18)$$

If the function, ψ, is to be finite everywhere, the function, f, must be zero at the origin and finite at infinity. Since at large distances from the origin the first term on the right hand side is the dominant one, the asymptotic form of "f" is exp(-x). If the substitution f = -f is made into the equation, it is unchanged. Therefore if "f" is a solution to the equation, "-f" is also a solution.

If a change of variables is made to

$$z = e^{-x} \qquad (6.19)$$

the equation is transformed to

$$\frac{d^2 f}{d z^2} + \frac{1}{z}\frac{df}{dz} = \frac{f}{z^2} - \frac{f^3}{3z^2(\log z)^2} \tag{6.20}$$

The boundary conditions are now that

$$f = 0\,, \quad z = 0$$
$$f = 0\,, \quad z = 1 \tag{6.21}$$

We also have that near the origin, "f" is linear in "z".

Eqn. (6.20) has been solved numerically on the LGP-30 computer at the University of Alberta. The solution was carried out by using a step by step forward integration. The following approximations were made for the values of the derivatives at any one point:

$$f''_n = \frac{f_{n+1} - 2f_n + f_{n-1}}{h^2}$$
$$f'_n = \frac{f_{n+1} - f_{n-1}}{2h} \tag{6.22}$$

Here "h" is the length of the interval between successive points. These formulas are substituted into eqn. (6.20) and the resulting equation is solved for f_{n+1}. The result is

$$f_{n+1} = \frac{f_{n-1}\left[\frac{1}{2hz_n} - \frac{1}{h^2}\right] + f_n\left[\frac{1}{z_n} - \frac{f_n^{\,2}}{3z_n^2(\log z_n)^2} + \frac{2z_n^2}{h^2}\right]}{\left[\frac{1}{h^2} + \frac{1}{2hz_n}\right]} \tag{6.23}$$

To find the value of the function at any one point by this method, the value of the function at two preceeding points must be known. The function was set equal to zero at the origin and the value of the function at the first point was set arbitrarily. Then the value of the function was calculated at each point with the use of the above formula, up to the point $z = 1$. Since the boundary condition requires that the function be zero at this point, the initial value was varied until the final value became zero.

By this method, three solutions have been obtained for eqn. (6.20) which fit the boundary conditions. The solutions are characterized by the number of nodes which they contain. Thus the first solution $\varphi_0 = f_0/x$ has no nodes, φ_1 , has one node and φ_2 has two nodes. Presumably there are many more solutions; perhaps an infinite number. More solutions could not be obtained because of the limited accuracy of the machine.

The functions have been calculated with 16, 128, and 512 points in the interval between 0 and 1. Tables I and II give "f" as a function of "z" as calculated on the computer. Only 32 points are listed for each solution.

z	16-point solution	128-point solution	512-point solution
0.00000	0.000000	0.000000	0.000000
0.03125		0.146789	0.146917
0.06250	0.287915	0.293504	0.293757
0.09375		0.439997	0.440370
0.12500	0.575140	0.586038	0.586523
0.15625		0.731300	0.731883
0.18750	0.859569	0.875343	0.876012
0.21875		1.017598	1.018338
0.25000	1.137402	1.157341	1.158129
0.28125		1.293676	1.294488
0.31250	1.402504	1.425507	1.426316
0.34375		1.551521	1.552297
0.37500	1.645690	1.670168	1.670884
0.40625		1.779652	1.780275
0.43750	1.854061	1.877928	1.878421
0.46875		1.962723	1.963051
0.50000	2.010752	2.031570	2.031695
0.53125		2.081874	2.081761
0.56250	2.095632	2.111011	2.110628
0.59375		2.116463	2.115789
0.62599	2.087702	2.095976	2.095005
0.65625		2.047747	2.046493
0.68750	1.969645	1.970621	1.969127
0.71875		1.864271	1.862607
0.75000	1.734007	1.729344	1.727586
0.78125		1.567532	1.565764
0.81250	1.388784	1.381567	1.379875
0.84375		1.175114	1.173579
0.87500	0.959046	0.952576	0.951270
0.90625		0.718831	0.717815
0.93750	0.482435	0.478931	0.478246
0.96875		0.237806	0.237473
1.00000	0.000000	0.000000	-0.000021

TABLE I

FIRST SOLUTION OF EQUATION (6.20)

z	f_1 128-point	f_2 128-point
0.00000	0.000000	0.000000
0.03125	0.922109	4.316718
0.06250	1.826049	7.001103
0.09375	2.677376	7.211515
0.12500	3.429140	5.971093
0.18750	4.030550	4.398816
0.18750	4.439408	2.954004
0.21875	4.634133	1.721698
0.25000	4.619803	0.670752
0.28125	4.425375	-0.247380
0.31250	4.094164	-1.071010
0.34375	3.672684	-1.822504
0.37500	3.202162	-2.507862
0.40625	2.714333	-3.117395
0.43750	2.230828	-3.627526
0.46875	1.764585	-4.005116
0.50000	1.321993	-4.215533
0.53125	0.904953	-4.233714
0.56250	0.512582	-4.054591
0.59375	0.142488	-3.697656
0.62500	-0.208254	-3.202726
0.65625	-0.542297	-2.618971
0.68750	-0.861185	-1.992823
0.71875	-1.164283	-1.395654
0.75000	-1.447212	-0.740834
0.78125	-1.699488	-0.145099
0.81250	-1.901250	0.427225
0.84375	-2.019615	0.976756
0.87500	-2.007210	1.489378
0.90625	-1.809018	1.897868
0.93750	-1.385759	1.999677
0.96875	-0.752024	1.407735
1.00000	-0.000001	0.000006

TABLE II

SECOND AND THIRD SOLUTIONS OF EQUATION (6.20)

Figures I - VI are graphs of φ and "f" as functions of "x".

When the number of intervals is increased, the value of the function tends to increase near the origin and decrease near the point $z = 1$. The first solution is the only one which has been calculated at all three intervals. The maximum percentage difference between the 16-point solution and the 128-point solution is about 2%, while that between the 128-point solution and 512-point solution is about 0.1%. The 128-point solutions were used in all the calculations.

Maxwell's equation for the static field is

$$\operatorname{div} \vec{E} = 4\pi\rho$$

where ρ is the charge density. The right hand side of eqn. (6.1) is identified with the charge density for the particle

$$-4\pi\rho = \frac{m^2c^2}{\hbar^2}\varphi - \frac{g^2}{\hbar^2c^2}\frac{\varphi^3}{3} \tag{6.24}$$

The total charge of the particle, Q, is

$$Q = \int \rho\, dV = -\frac{1}{4\pi}\int \nabla^2\varphi\, dV$$

Using the divergence theorem we have

$$Q = -\frac{1}{4\pi}\int \nabla\varphi\cdot d\vec{S}$$

Since φ goes exponentially at large distances from the origin, the surface integral over a sphere goes to zero as the radius of the sphere goes to infinity. Therefore the total charge of the particle is zero and

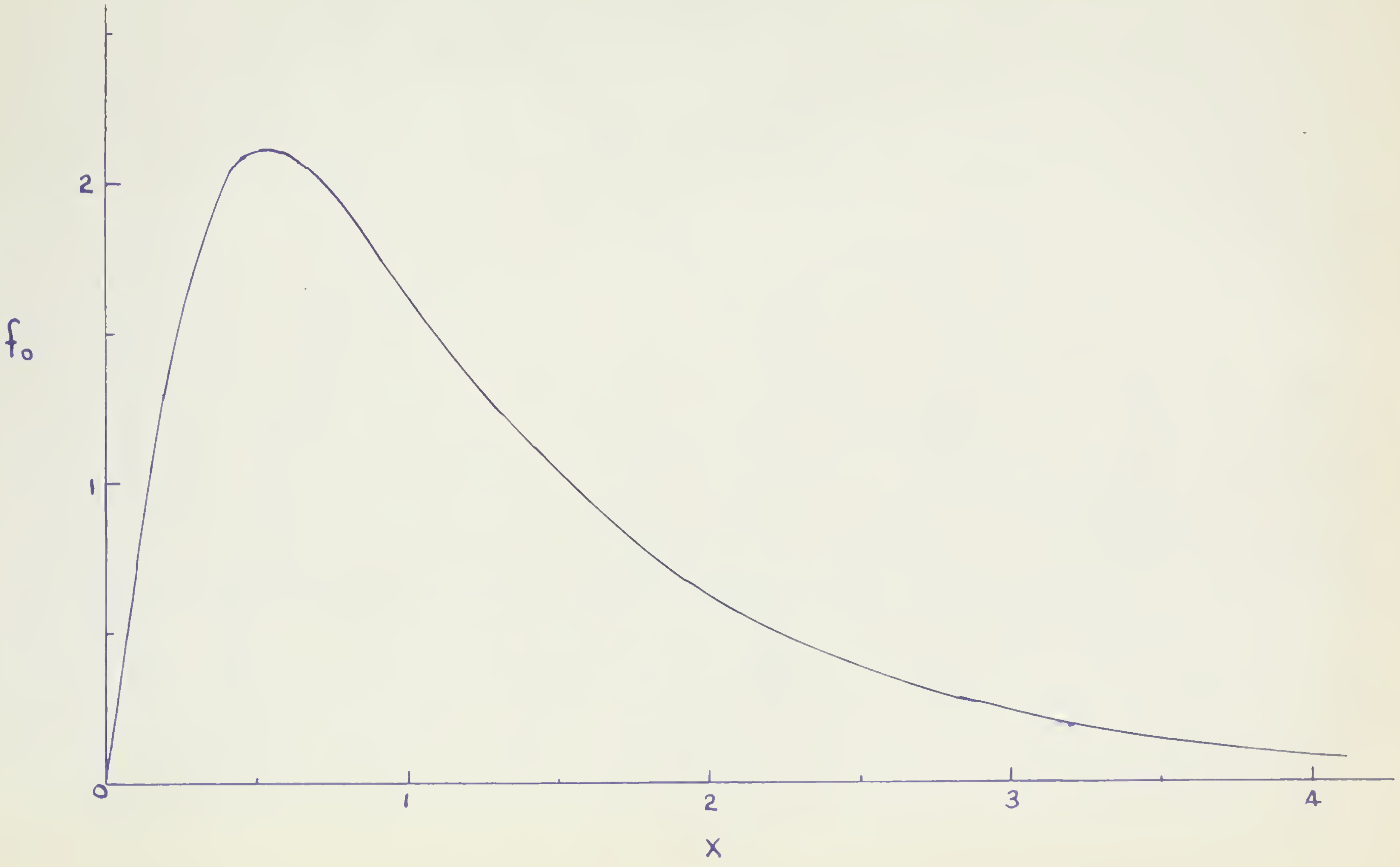


Fig. 1

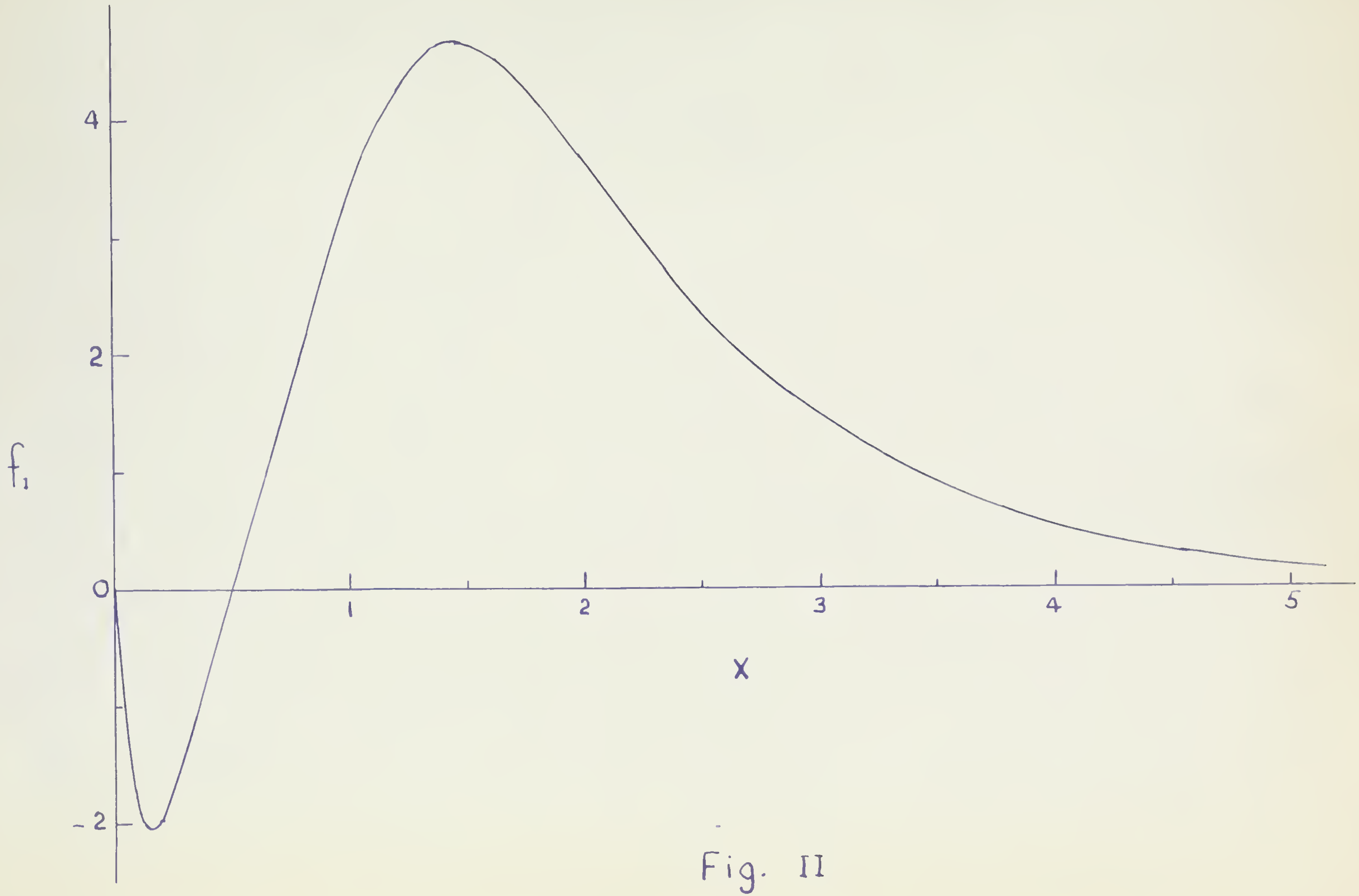


Fig. II

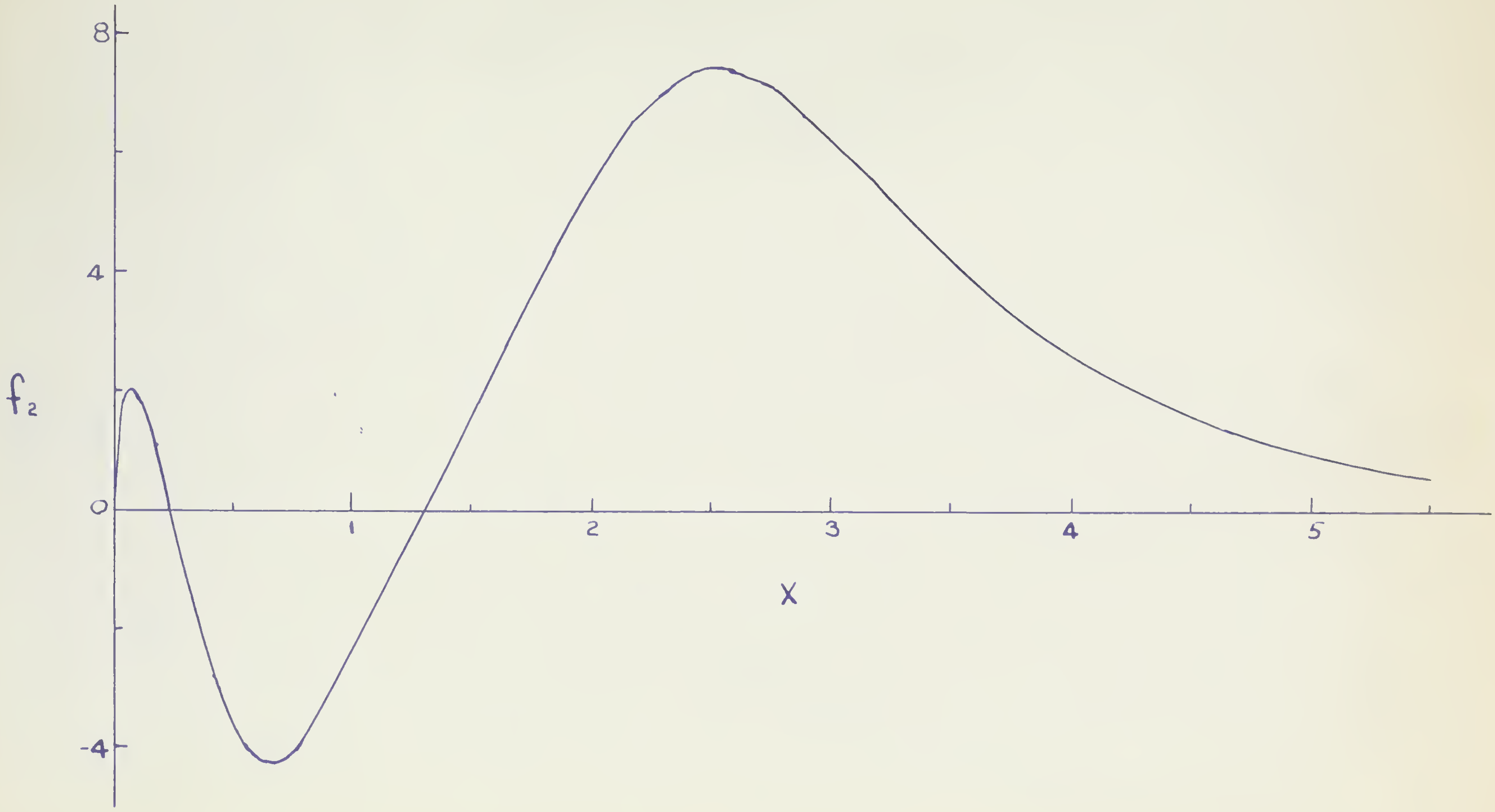
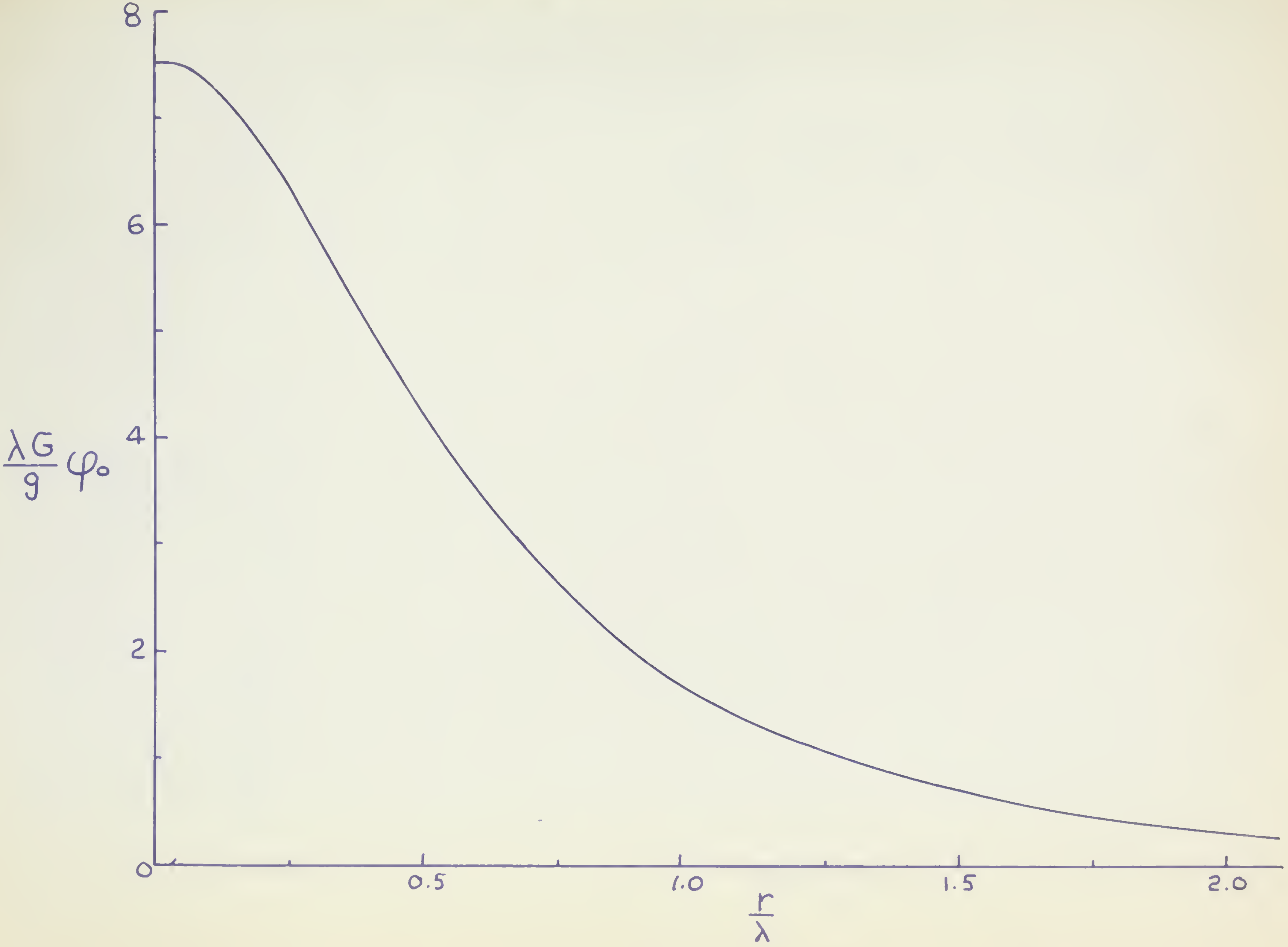


Fig. III


$\frac{\lambda G}{g}\varphi_0$
8
6
4
2
0
0.5
1.0
1.5
2.0
$\frac{r}{\lambda}$

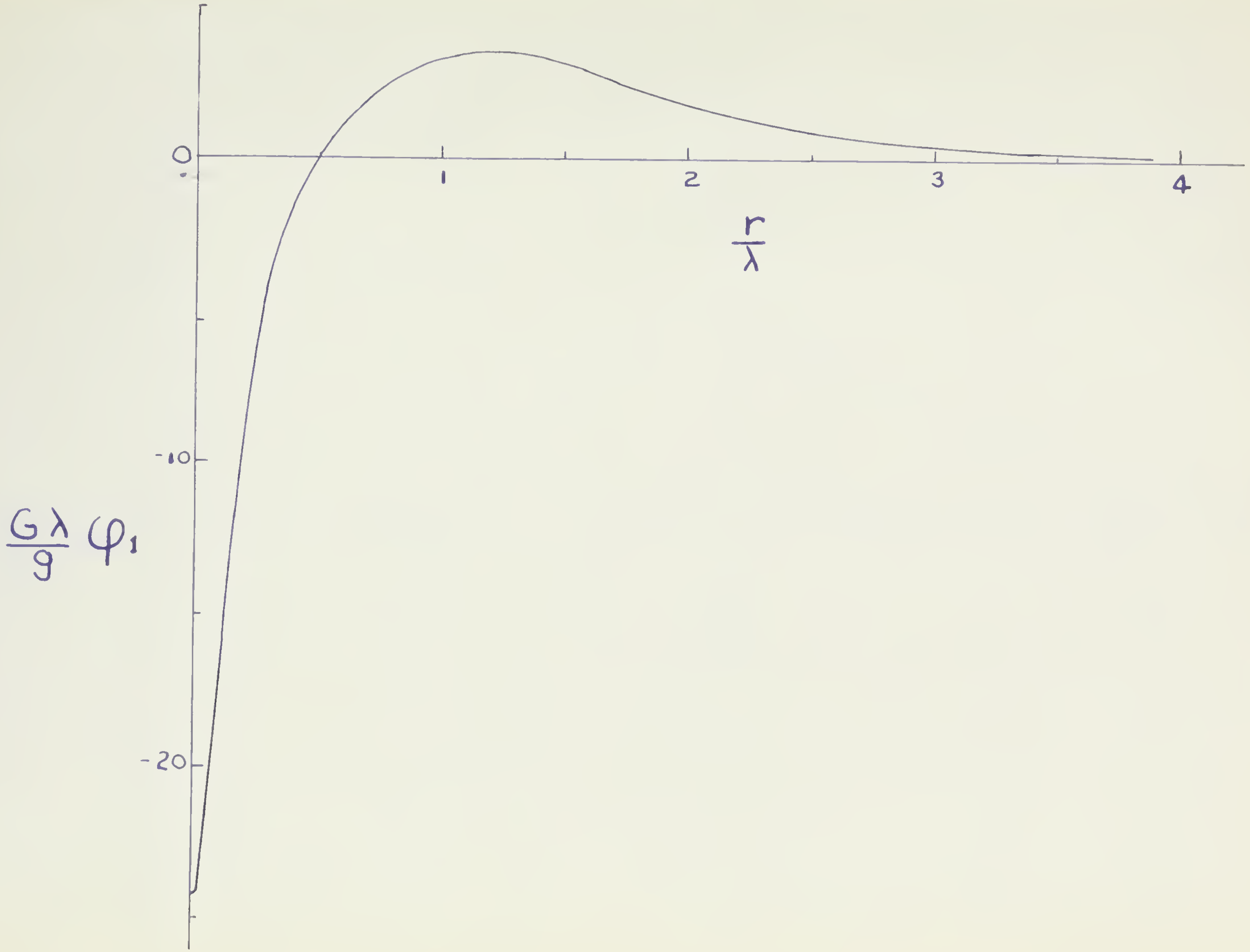


Fig V

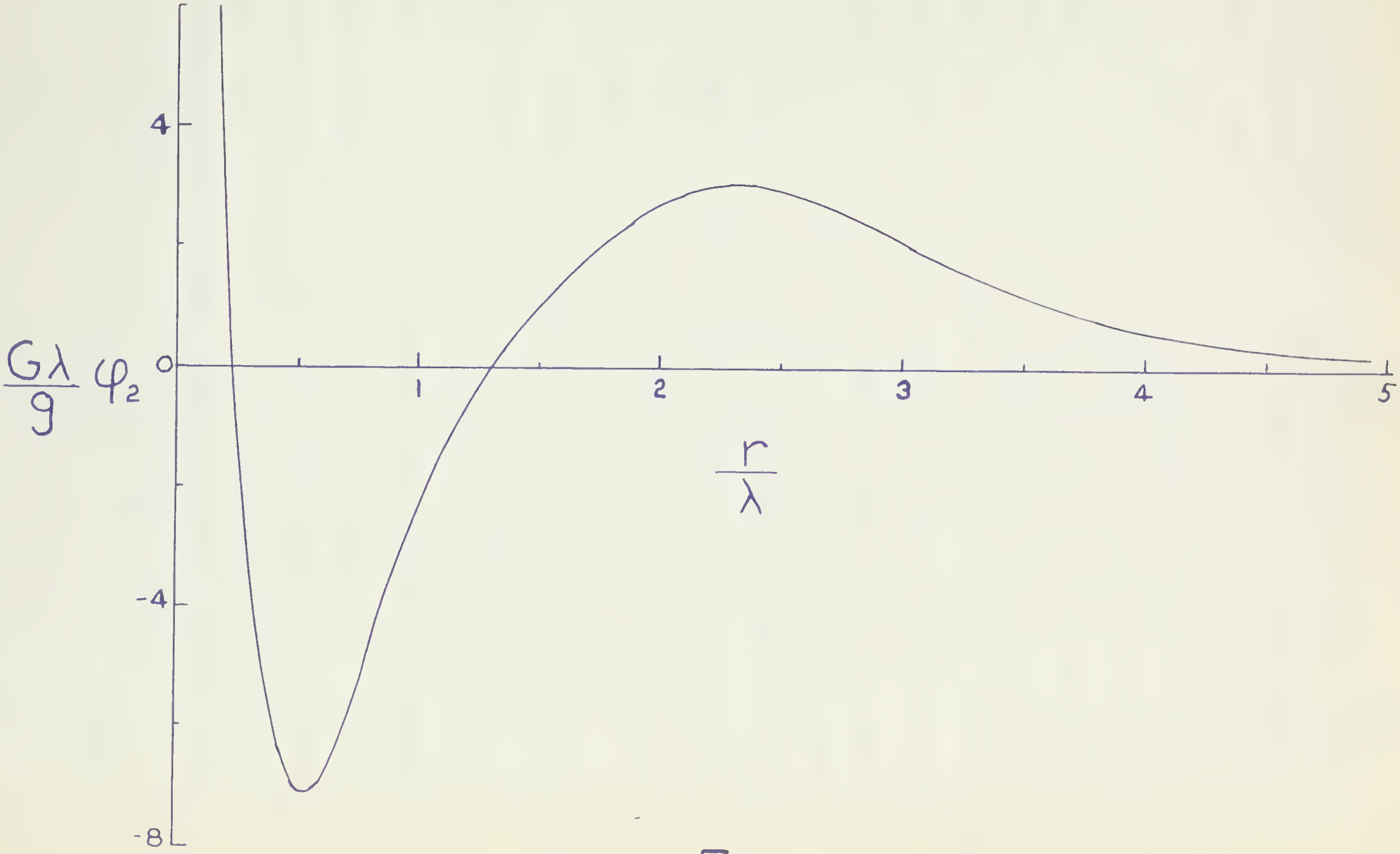


Fig. VI

this equation must describe a neutral particle. Since we are dealing with a static case, no intrinsic magnetic moment or spin is involved.

The potential for a charged particle would have the form $\varphi = A + q/r$ asymptotically, where "A" is a constant and "q" is the total charge of the particle. No solution of eqn. (6.1) has this asymptotic behaviour. Thus a spherically symmetric solution of this equation could not describe a charged particle. However, since charged particles are described quantum mechanically by complex fields, it is possible that a description of a charged particle could be obtained in our static case by carrying out a gauge transformation, in which case according to eqn. (5.12), E would be replaced by the complex function $\vec{E}\exp(igS/\hbar c)$.

Figures VII - IX are plots of ρ and $r^2\rho$ as functions of the radial distance, r, for the three solutions. The graphs of the charge density for the higher solutions exhibit an orbital arrangement with bands of concentrated charge separated by regions with little charge. If more solutions exist for this differential equation, one would expect them to show more of these charge bands. The first solution has two of these charge bands of opposite sign. The total charge in one of these hands, q, is

$$q = 3.016\ g/G \qquad (6.25)$$

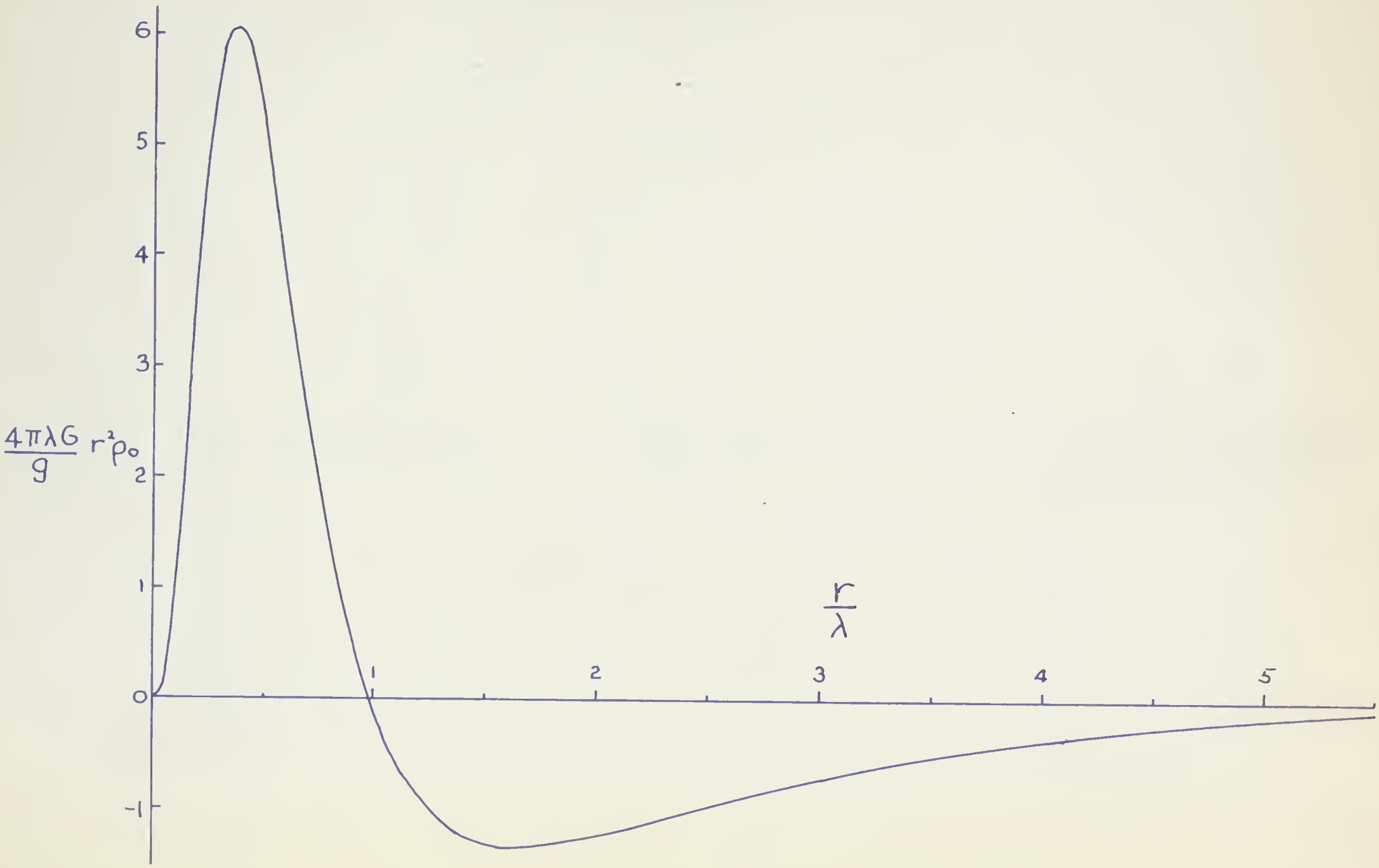


Fig. VII

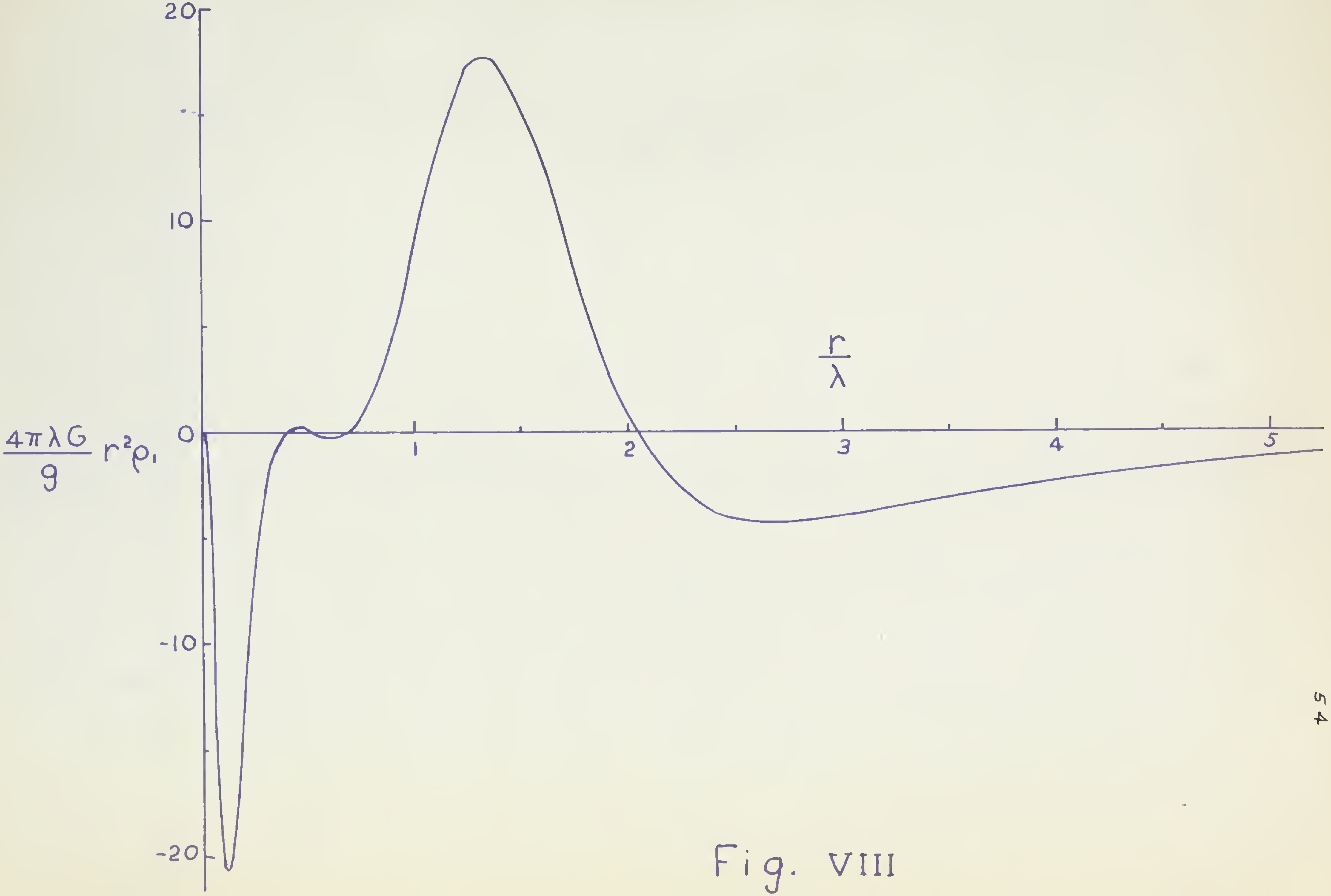


Fig. VIII

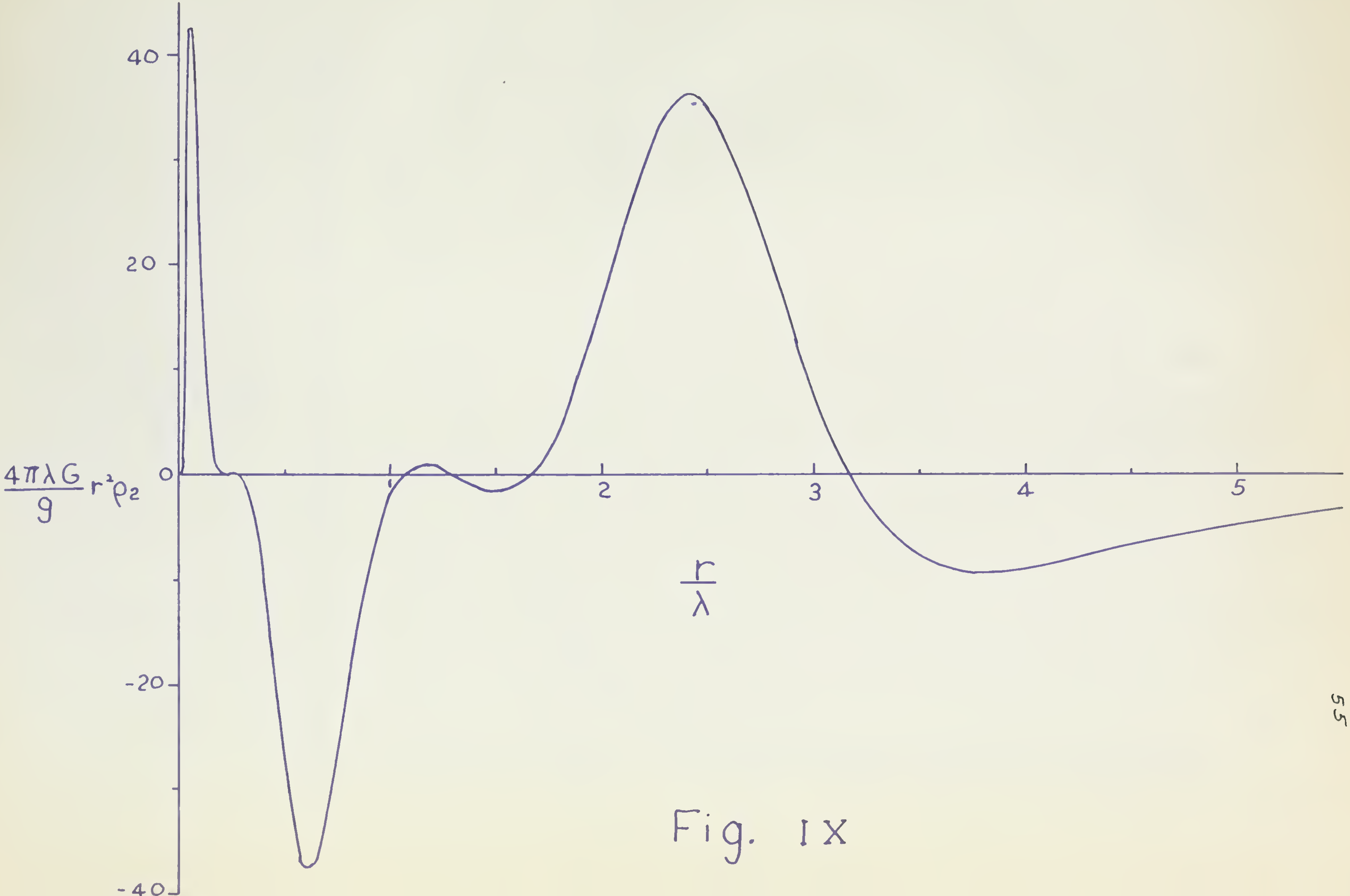


Fig. IX

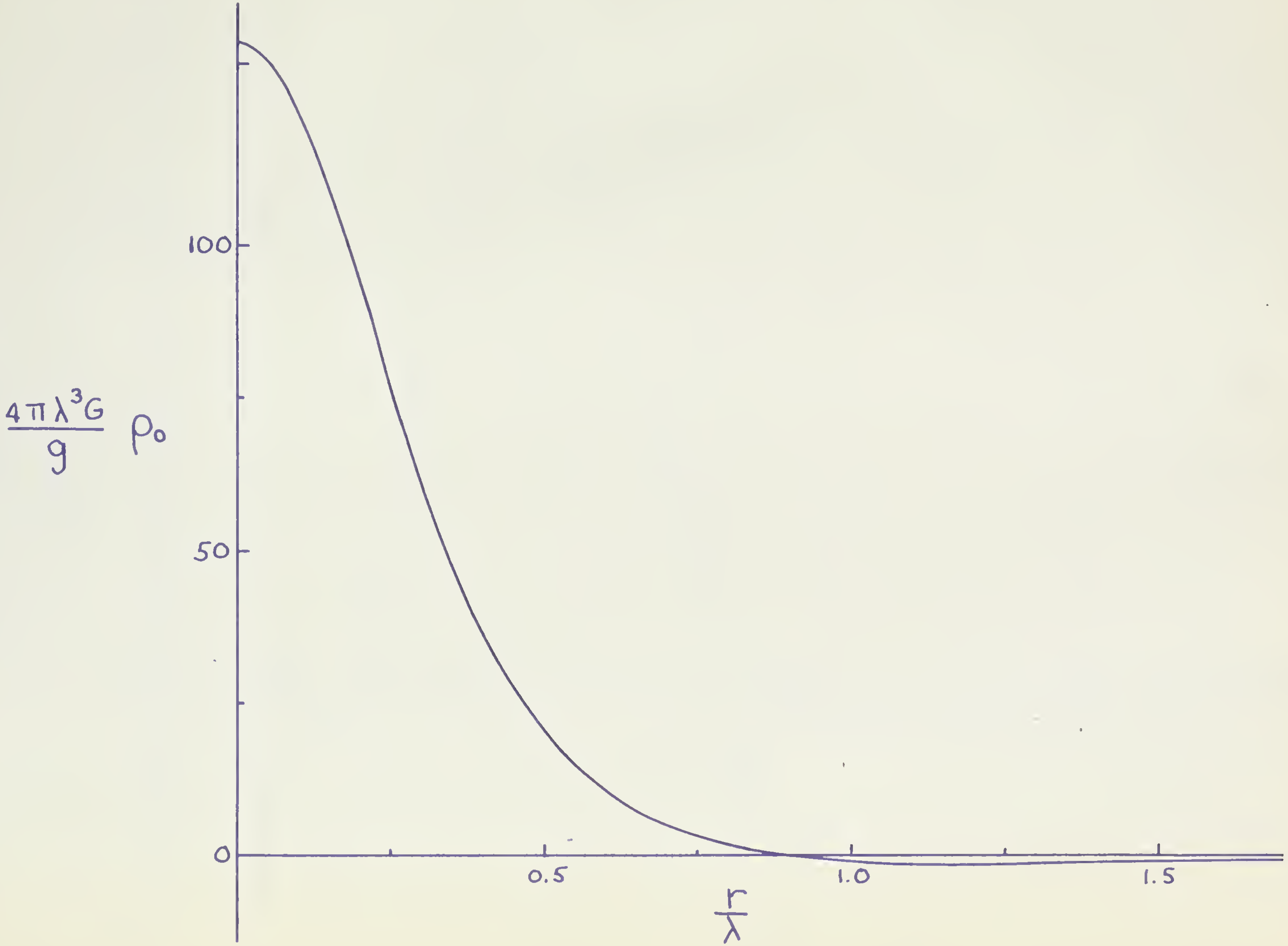


Fig. X

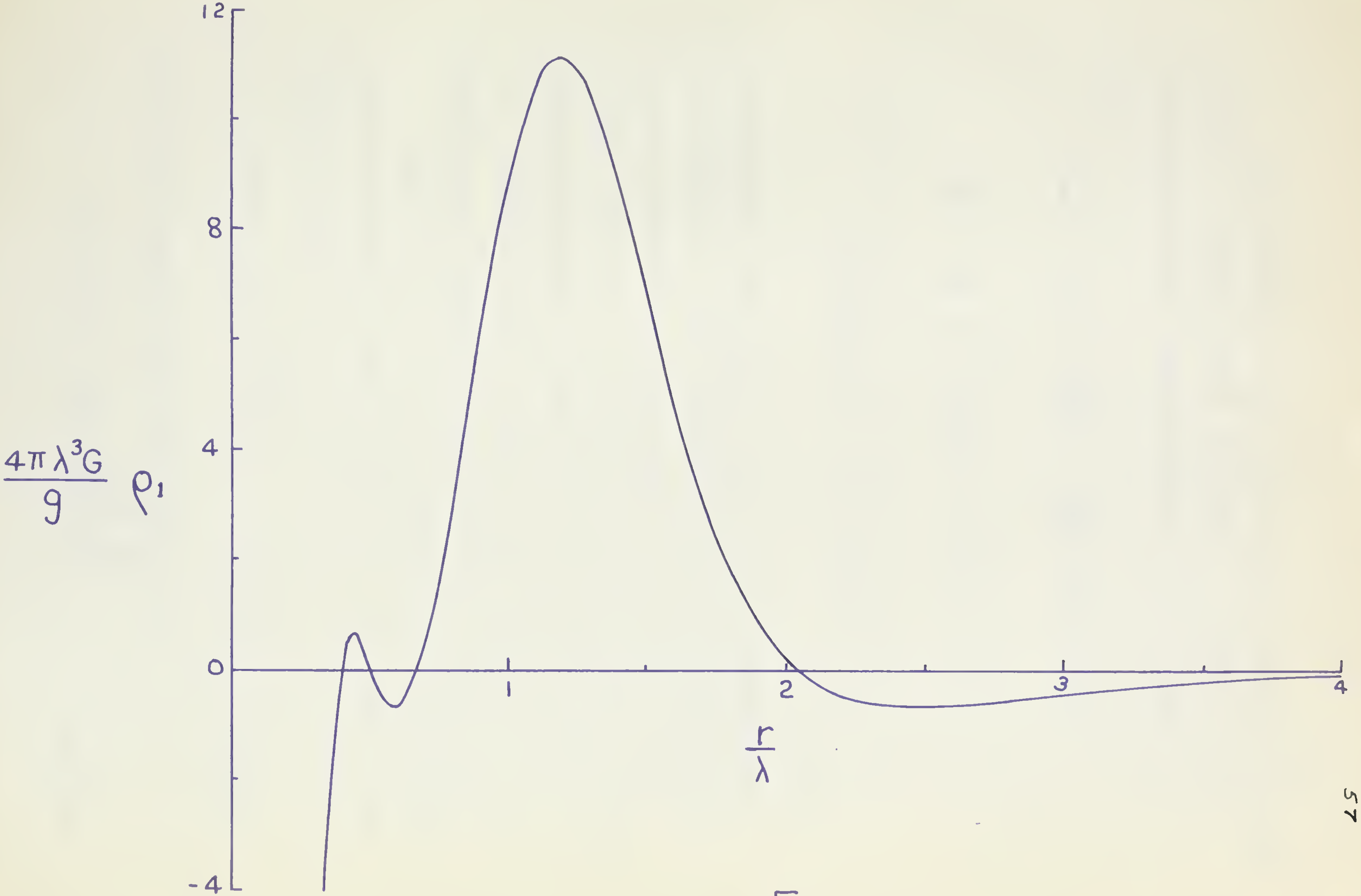


Fig. XI

The accuracy of the first machine solution has been checked by various means. The first criterion is that the integral of the charge density over all space is zero.

$$Q = -\frac{1}{4\pi}\int\left(\frac{m^2c^2}{\hbar^2}\psi - \frac{g^2}{\hbar^2c^2}\frac{\psi^3}{3}\right)dV = 0$$

$$Q = \frac{g}{G}\int_0^\infty\left(\frac{\psi^3}{3} - \psi\right)x^2\,dx = 0$$

$$\int_0^\infty \frac{\psi^3}{3}x^2\,dx = \int_0^\infty \psi\, x^2\,dx \tag{6.26}$$

The integrals are given in Table IV. The difference in the integrals for the 16-point solution is about 0.09%, and the difference for the 128-point solution is about 0.006%. This seems to indicate that the machine solutions are quite accurate, at least for the first solution.

Another attempt to obtain some indication of the existence and accuracy of the solutions is given below.

Equation (6.11) can be treated as an inhomogenous linear differential equation with the cubic term as the source function.

$$\nabla^2\psi - \psi = -P(x) \qquad P = \frac{\psi^3}{3} \tag{6.27}$$

The Green's function for this equation is

$$G(\vec{x},\vec{y}) = \frac{1}{4\pi}\,\frac{e^{-|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \tag{6.28}$$

Therefore

$$\Psi = \frac{1}{4\pi} \int \frac{e^{-|\vec{y}-\vec{x}|}}{|\vec{y}-\vec{x}|} P(y)\, d\vec{y}$$

$$\Psi = \frac{1}{2} \int_0^\infty y^2 dy \int_0^\pi \sin\theta\, d\theta\, \frac{e^{-|\vec{y}-\vec{x}|}}{|\vec{y}-\vec{x}|} P(y)$$

Integration with respect to the polar angle gives

$$\Psi = \frac{e^{-x}}{x} \int_0^x y\, P(y) \sinh y\, dy + \frac{\sinh x}{x} \int_x^\infty y\, P(y)\, e^{-y} dy$$

$$\Psi_G = \frac{e^{-x}}{x} \int_0^x y\, \frac{\Psi_t^{\,3}}{3} \sinh y\, dy + \frac{\sinh x}{x} \int_x^\infty y\, \frac{\Psi_t^{\,3}}{3} e^{-y} dy$$

$$f_G = e^{-x} \int_0^x \frac{f_t^{\,3}}{3y^2} \sinh y\, dy + \sinh x \int_x^\infty \frac{f_t^{\,3}}{3y^2} e^{-y} dy \tag{6.29}$$

An analytic approximation was made for the function f_o. This approximation was of the form

$$f_t = A(e^{-x} - e^{-bx})$$

$$A = 4.7, \quad b = 3.45 \tag{6.30}$$

If this is substituted into eqn. (6.29), the result is

$$\begin{aligned} \frac{6 f_G}{A^3} = {} & (0.266574) e^{-x} \\ & + e^{-x}\left[2\, Ei(2x) - 13.35\, Ei(4.45x) + 20.7\, Ei(6.9x) \right. \\ & \qquad \left. - 9.35\, Ei(9.35x)\right] \qquad (6.31) \\ & - e^{-x}\left[4\, Ei(4x) - 19.35\, Ei(6.45x) + 26.7\, Ei(8.9x) \right. \\ & \qquad \left. - 11.35\, Ei(11.35x)\right] \end{aligned}$$

where (6.32)

$$Ei(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt$$

The functions f_t and f_G are tabulated in Table III.

An improved approximation was made of the form

$$f_t = A\left(e^{-x} - e^{-bx} - a x e^{-cx}\right)$$

$$A = 4.7\,, \quad b = 4\,, \quad a = 1.4\,, \quad c = 7 \tag{6.33}$$

The result after substitution into eqn. (6.29) is

$$\begin{aligned} \frac{6 f_G}{A^3} = {} & (0.280320)\,e^{-x} - (0.0525)\,e^{-15x} + (0.036409)\,e^{-18x} \\ & + (0.0011406 - 0.012473x)\,e^{-21x} \\ & + e^{-x}\big[2\,Ei(2x) - 15\,Ei(5x) + 28.2\,Ei(8x) \\ & \qquad - 19.4\,Ei(11x) + 4.2\,Ei(14x)\big] \\ & - e^{x}\big[4\,Ei(4x) - 21\,Ei(7x) + 34.2\,Ei(10x) \\ & \qquad - 21.4\,Ei(13x) + 4.2\,Ei(16x)\big] \end{aligned} \tag{6.34}$$

These functions are tabulated in Table III.

x	f_o computer	FIRST APPROXIMATION		SECOND APPROXIMATION	
		f_t	f_G	f_t	f_G
0.00	0.0000	0.000	0.000	0.0000	0.0000
0.10	0.7296	0.924	0.885	0.7755	0.6792
0.22	1.4434	1.572	1.630	1.5120	1.5052
0.36	1.9392	1.920	2.055	1.9749	1.9899
0.52	2.1163	2.013	2.156	2.1173	2.1439
0.70	2.0249	1.914	2.024	2.0138	2.0363
0.92	1.7535	1.676	1.744	1.7448	1.7570
1.20	1.3784	1.341	1.364	1.3752	1.3789
1.60	0.9414	0.930	0.930	0.9410	0.9409
2.30	0.4705	0.470	0.465	0.4707	0.4702

TABLE III

TEST OF FIRST SOLUTION WITH GREEN'S FUNCTION

It was found that the maximum deviation between the analytic approximation, (6.30), and the Green's function solution is about 7%. With the improved approximation, (6.33), the maximum deviation is about 1.3%. These results suggest that even better analytic approximations to the machine solution would probably show even less deviations from the Green's function solution.

Since a Lagrangian has not been obtained for eqn. (5.13), the energy-momentum tensor cannot be determined. Hence the total energy of the particle cannot be evaluated. The assumption is made that the total energy of the system can be taken as the space integral of the product of the charge density and the potential, i.e.

$$\mathcal{E} = \int \rho \, \varphi \, dV \tag{6.35}$$

where $\mathcal{E}$ is the total energy. From eqn. (6.24) we have

$$\begin{aligned} \mathcal{E} &= -\frac{1}{4\pi} \int \left(\frac{\varphi}{\lambda^2} - \frac{g^2}{\hbar^2 c^2} \frac{\varphi^3}{3} \right) \varphi \, dV \\ &= \frac{mc^2}{G} \int_0^\infty \left(\frac{\psi^4}{3} - \psi^2 \right) x^2 \, dx \\ &= \frac{mc^2}{G} \left[\frac{I_4}{3} - I_2 \right] \end{aligned} \tag{6.36}$$

$$I_4 = \int_0^\infty \psi^4 x^2 \, dx \,, \quad I_2 = \int_0^\infty \psi^2 x^2 \, dx$$

These integrals have been calculated for the three solutions and have been tabulated in Table IV.

CHARGE INTEGRALS FOR FIRST SOLUTION

	$\int_0^\infty \psi x^2 dx$	$\int_0^\infty \frac{\psi^3}{3} x^2 dx$
16-point solution	4.29697	4.30075
128-point solution	4.366975	4.366726

ENERGY INTEGRALS

	$\int_0^\infty \psi^2 x^2 dx$	$\int_0^\infty \psi^4 x^2 dx$
16-point first solution	4.422716	53.67182
128-point first solution	4.509968	54.12047
128-point second solution	28.36375	340.6186
128-point third solution	85.97097	1031.239

TABLE IV

It is interesting to note that $I_4 = 12I_2$ for each of the three solutions obtained. Thus if the energy were some linear combination of the two integrals, different from (6.36), the ratio of the energies for the three particles would be unchanged.

For the first solution we have

$$I_2 = 4.509968$$

$$\frac{I_4}{3} = 18.04016$$

This gives a value for the energy of the first particle,

$$\mathcal{E}_0 = 13.53 \frac{mc^2}{G} \tag{6.37}$$

If the mass "m" in the equations is to be taken as that of the lowest state, i.e. the first particle, the energy from the first solution must be identically equal to mc^2. This gives a value of 13.53 for the coupling constant, $g^2/\hbar c$, which is of the order of the coupling constant for the "strong" interactions.

Since the value of the coupling constant for electromagnetic interactions, α , is known, the ratio of "g", the basic unit of charge in this system, to "e" the charge on the electron can be calculated. We get

$$\frac{g^2}{e^2} = 137 \times 13.53 = 1852 \qquad (6.38)$$

and

$$\frac{g}{e} = 43.06$$

It is interesting to note that g^2/e^2 is within one per cent of the nucleon electron mass ratio.

We had earlier in eqn. (6.25) that the charge in one of the bands in the first solution, q, is 3.106 g/G. This charge is then equal to

$$q = 0.2229\ g = 9.60\ e \qquad (6.39)$$

The energies of the second and third particles with respect to the first work out to be

$$\mathcal{E}_1 = 6.293\ \mathcal{E}_0 \qquad (6.40)$$

$$\mathcal{E}_2 = 19.06\ \mathcal{E}_0 \qquad (6.41)$$

Interaction potentials between the particles have been calculated using analytic approximations to the machine solutions. We take the interaction potential V_{ij}, between two particles as

$$V_{ij}(r) = \frac{1}{2}\left[\int \varphi_i(R)\,\rho_j(r')\,d\vec{r}\,' + \int \varphi_j(R)\,\rho_i(r')\,d\vec{r}\,'\right] \qquad (6.42)$$

$$R = |\vec{r} - \vec{r}\,'|$$

Using eqn. (6.24) and the divergence theorem, we obtain the following result:

$$\begin{aligned}\int \varphi_i \rho_j \, d\vec{r}' &= -\frac{1}{4\pi}\int \varphi_i \nabla^2 \varphi_j \, d\vec{r}' \\ &= \frac{1}{4\pi}\int \nabla\varphi_i \cdot \nabla\varphi_j \, d\vec{r}' \\ &= -\frac{1}{4\pi}\int \varphi_j \nabla^2 \varphi_i \, d\vec{r}' \\ &= \int \varphi_j \rho_i \, d\vec{r}'\end{aligned}$$

Therefore

$$V_{ij} = V_{ji} = \int \varphi_i (|\vec{r} - \vec{r}'|) \, \rho_j(r') \, d\vec{r}' \qquad (6.43)$$

Since for a given solution φ_i, $-\varphi_i$ is also a solution, it follows that the interaction potential can be $\pm V_{ij}$.

In calculating the interaction potential, we have assumed that the charge distribution of one particle is unchanged in the field of another particle. Since there will be mutual distortion when the particles are close together, the interaction potential is only qualitatively correct near the origin. In doing the calculations, it was found that the interaction potential could not be determined accurately near the origin. There would be a further modification if the two particles were in relative motion since the interaction is velocity dependent.

The interaction potentials have been calculated between two particles of the first kind and between a

particle of the first kind and one of the second kind. This was done by making analytic approximations for φ_0, φ_1 and ρ_0 as calculated by the computer. Interaction potentials involving the third particle or the one between two particles of the second kind could not be determined accurately because of the difficulty of approximating the potentials and charge distributions sufficiently closely.

The approximations taken were of the form

$$\varphi_0 = \frac{g}{\lambda G}\, 4.7 \left(\frac{e^{-x} - e^{-3.45x}}{x}\right) \tag{6.44}$$

$$\rho_0 = -\frac{1}{4\pi}\frac{g}{\lambda^3 G}\, 4.7 \left(\frac{e^{-x} - 26e^{-4x} + 25e^{-6.5x}}{x}\right) \tag{6.45}$$

$$\varphi_1 = \frac{g}{\lambda G}\, 29.6 \left(\frac{e^{-x} - 6e^{-2.5x} + 5e^{-3x}}{x}\right) \tag{6.46}$$

$$V_{00} = \frac{mc^2}{G}\left[\left(\frac{26.99331}{x} - 11.045\right)e^{-x} - 113.453\,\frac{e^{-3.45x}}{x} + 91.066\,\frac{e^{-4x}}{x} - 4.6066\,\frac{e^{-6.5x}}{x}\right] \tag{6.47}$$

$$V_{01} = \frac{mc^2}{G}\left[\left(\frac{228.8705}{x} - 69.56\right)e^{-x} - 1805.25\,\frac{e^{-2.5x}}{x} + 2147.7\,\frac{e^{-3x}}{x} - 598.98\,\frac{e^{-4x}}{x} + 27.65\,\frac{e^{-6.5x}}{x}\right] \tag{6.48}$$

These potentials are tabulated in Table V and plotted in Figures XII and XIII.

From eqn. (6.43), we get the value of the interaction potential at the origin to be

$$V_{ij}(0) = \int \varphi_i(r)\,\rho_j(r)\,d\vec{r} \tag{6.49}$$

If the two particles are of the same kind, we see from the definition of the total energy, eqn. (6.35), that the interaction potential at the origin is equal to the total energy of one of the particles. Thus we have from eqn. (6.37)

$$V_{00}(0) = 13.53\ \frac{mc^2}{G} \tag{6.50}$$

The value of $V_{01}(0)$ has been evaluated by numerical integration of the functions from the computer. This is

$$V_{01}(0) = 12.438\ \frac{mc^2}{G} \tag{6.51}$$

We see that these values are quite close to those obtained using the analytic approximations.

We note again that $-V_{00}$ and $-V_{01}$ are also possible interaction potentials, since the sign of the interaction potentials depend on the relative signs of the individual particles. Thus, for example, V_{00} can be essentially attractive as well as repulsive. Therefore V_{00} in Figure XII is shown to be essentially repulsive because the signs for the potentials of the interacting particles were the same. Whereas if they had been taken as different, the interaction potential would have been essentially attractive. The same arguments apply for V_{01}.

If we choose the mass, m, to be the mass of the neutral pi-meson, and the coupling constant, G, equal to 13.53, the energy scale in the case of V_{00} would correspond to a value at the origin of 135 Mev, i.e. $m_{\pi}c^2$. In the case of V_{01}, the minimum would correspond to an energy of 64 Mev or $0.47\ m_{\pi}c^2$. In both cases of course, the distance, x, would be in the units of the pi-meson Compton wavelength, 1.4×10^{-13}cm.

The shape of V_{01} is rather interesting because it indicates the possibility of forming a bound state of particles one and two. The binding energy would lie between 0 and 64 Mev, so that the total energy of the bound system including the energy of the individual

x	V_{00}	V_{01}
0.0	13.38	12.36
0.1		11.54
0.2	12.39	9.36
0.3		6.47
0.4	10.14	3.40
0.5		0.54
0.6	7.63	-1.88
0.7		-3.76
0.8	5.46	-5.11
0.9		-5.97
1.0	3.74	-6.42
1.2	2.47	-6.40
1.4		-5.64
1.5	1.24	
1.6		-4.60
1.8		-3.52
2.0	0.284	-2.55
2.2		-1.75
2.5	- 0.028	-0.875
3.0	- 0.103	-0.089
3.5	- 0.101	-0.191
4.0	- 0.079	0.243
4.5	- 0.056	0.212
5.0		0.162
5.5		0.114
6.0		0.078

TABLE V

INTERACTION POTENTIALS

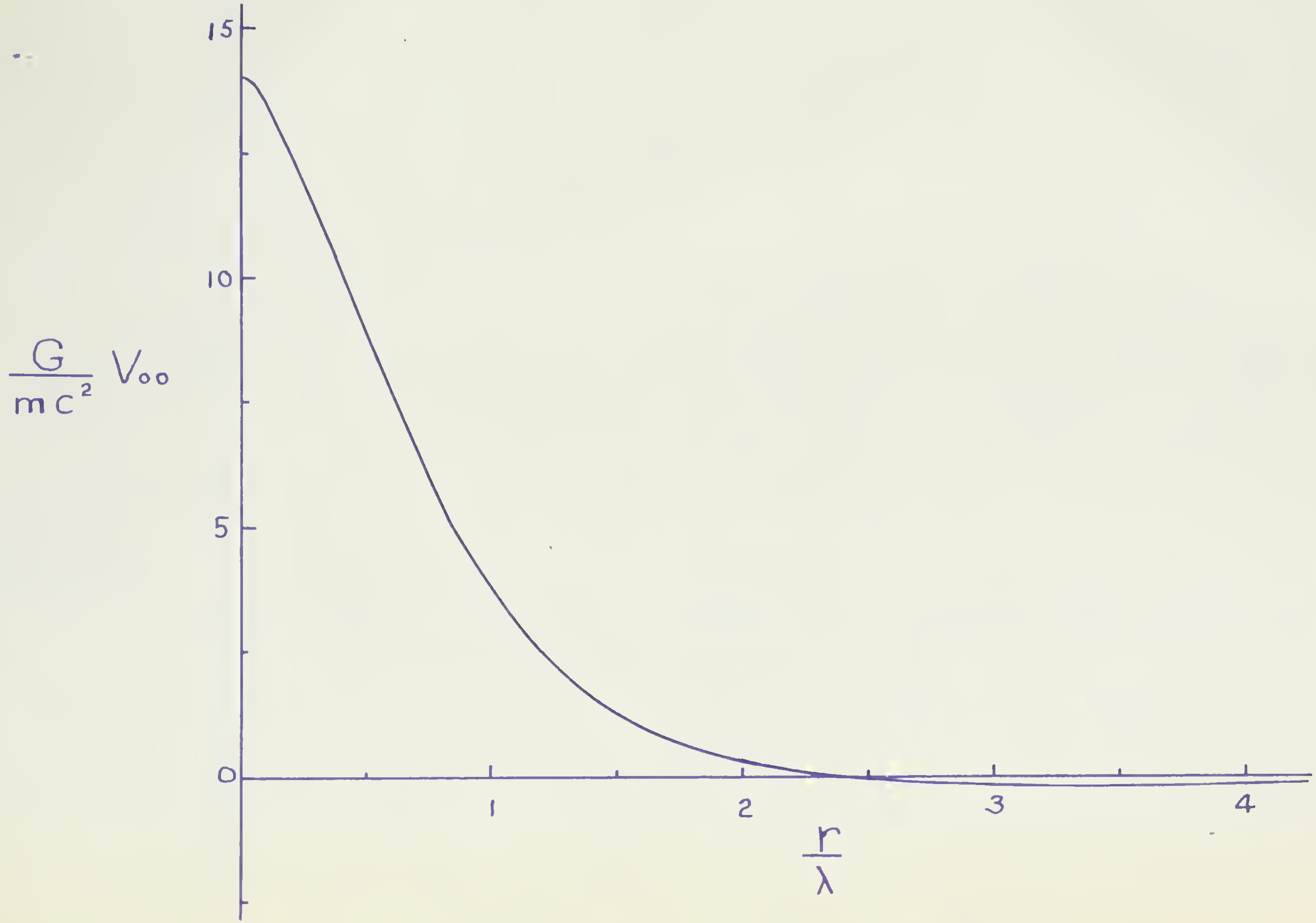


Fig. XII

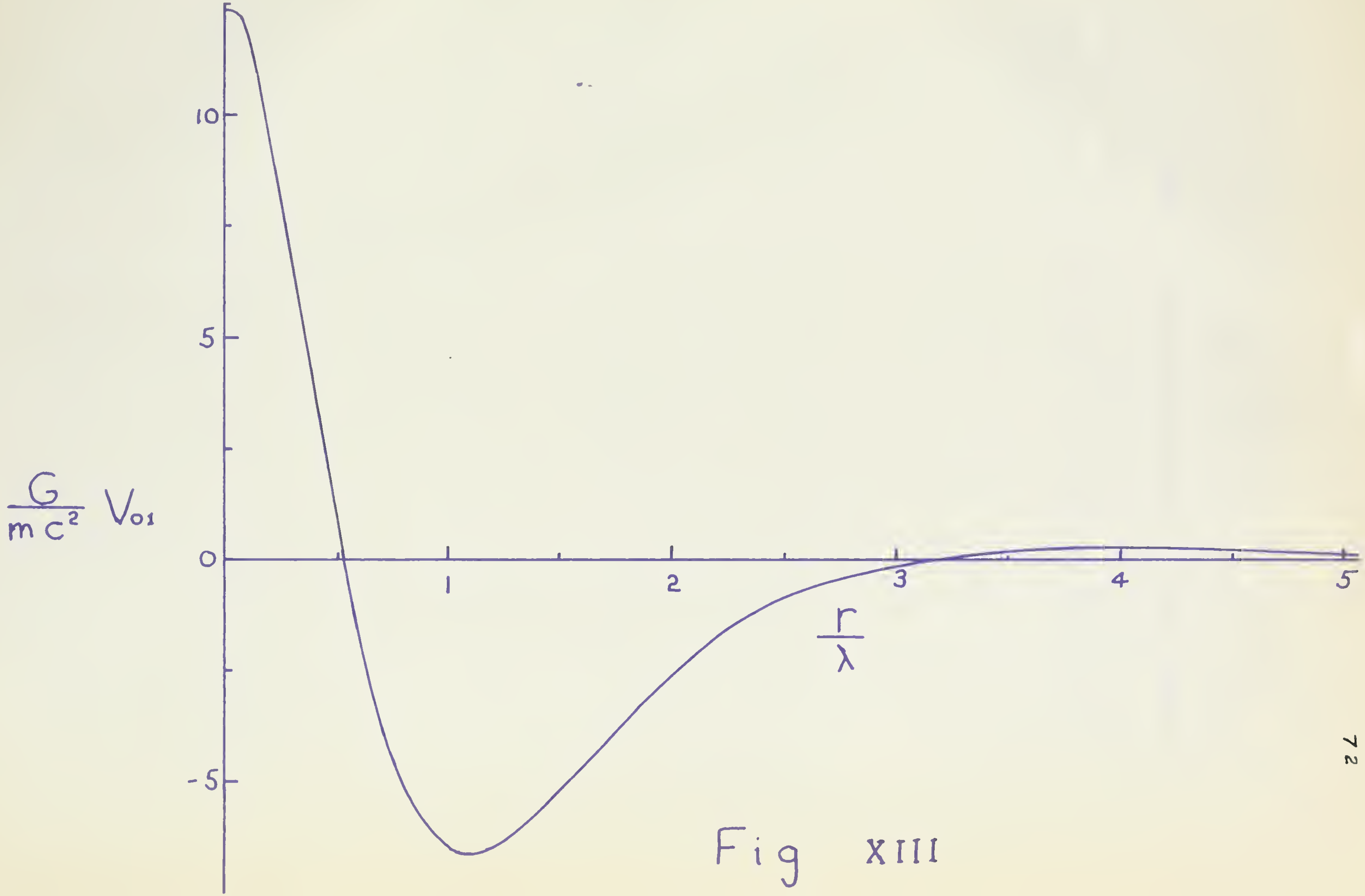


Fig XIII

particles would like in the range 6.8 to 7.3 $m_\pi c^2$. So that in our static case the mass ratio of the compound particle to that of the first particle is approximately the same as the ratio of nucleon mass to pi-meson mass. Needless to say, the spin of our compound particle is certainly not $\frac{1}{2}$.

VII. CONCLUSIONS

We have seen on the basis of a rather simple and perhaps naive static model, that it was possible to obtain a number of discrete states, which one may hope would represent in a more complete formulation, a spectrum of particles. Such a formulation would require an unambiguous definition for the energy-momentum tensor, which we feel could only stem from a first order boson equation.

With respect to the energy-momentum tensor, we would like to offer the following tentative suggestionto explain the fact as has already been mentioned on page 53, that for the three discrete solutions obtained, it was found that $\int \varphi^4 dV = 12 \int \varphi^2 dV$. Now we have not been able to show that such a relation is an automatic consequence of the static equation for φ , which leads us to believe that such a result may be of more fundamental significance. Consider, for example, a static Lagrangian of the form

$$\mathcal{L} = a \left[\frac{1}{2} \left(\frac{\partial \varphi}{\partial x_i} \right)^2 + \frac{1}{2} \varphi^2 - \frac{1}{12} \varphi^4 \right]$$

where "a" is an arbitrary constant. Disregarding as we must the relativistic invariance of the Lagrangian, the above Lagrangian nevertheless does yield the static

equation for φ . It is easy to show that

$$\int T_{11}\,dV = \int T_{22}\,dV = \int T_{33}\,dV = \int \left(\frac{\varphi^4}{12} - \varphi^2 \right) dV$$

which in our case is zero, and this is certainly a necessary requirement for a stationary particle. (T_{44} of course depends on the choice of "a"). In the time dependent case the symmetrization of the energy-momentum tensor may lead in the static limit to our expression for the energy, i.e. $\int \varphi \rho \, dV$, and to $\int (\frac{\varphi^4}{12} - \varphi^2) dV$ for $\int T_{\alpha\alpha}\,dV$ (α=1,2,3). Finally we note that in the case of our algebraic solution, the above considerations do not apply. Hence the requirement for particle-like solutions, such as $\int T_{\alpha\alpha}\,dV = 0$ may distinguish the physically acceptable solutions.

APPENDIX

F_{ik} - electromagnetic field tensor

$\vec{H}$ - magnetic field vector

$\vec{E}$ - electric field vector

$\vec{A}$ - vector potential

φ - scalar potential

$\vec{J}$ - current density

ρ - charge density

$$F_{ik} = \begin{pmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}$$

$$x_{1,2,3} = x, y, z \quad ; \quad x_4 = ict$$

$$j_{1,2,3} = J_{x,y,z} \quad ; \quad j_4 = ic\rho$$

$$A_{1,2,3} = A_{x,y,z} \quad ; \quad A_4 = i\varphi$$

$$d\tau = dx\, dy\, dz\, dict$$

$$ds^2 = dx_i^2$$

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